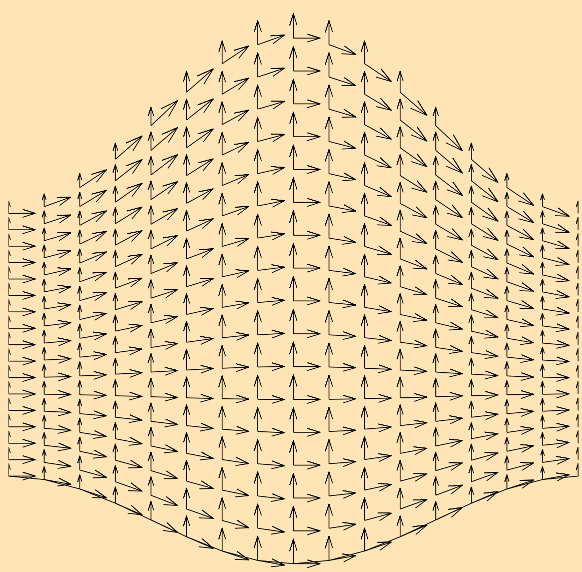
Tensors Unravelled

C. Pozrikidis



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Preface

A vector is a physical entity endowed with magnitude and orientation; examples are the position, the velocity, and the acceleration. A vector is typically described by its components in a chosen frame of reference, Cartesian or non-Cartesian, rectilinear or curvilinear.

A tensor is also a physical entity described by a higher number of components in a chosen frame of reference. In computational practice, the components of a tensor are typically stored in a two- or higher-dimensional array.

Vectors and tensors are distinguished by our ability to deduce their components in a certain frame of reference defined by a base from those in any other frame of reference defined by another base by simple geometrical transformations. To indicate this ability, we say that the physical entity represented by a tensor is objective or frame invariant.

All physical entities should be frame invariant; if they were not, observation and computation would be subjective, that is, the results would depend on the position and orientation of an observer or measuring instrument.

A zeroth-order tensor is a scalar whose value is frame-independent; examples are the temperature of a star, the angle between two vectors, and the distance between two cities. A first-order tensor is a vector that has the same magnitude and points in the same direction independent of the location of an observer.

The stress tensor is a second-order tensor encapsulating the tractions exerted on three small mutually perpendicular faces in a solid or fluid. Higher-order tensors and their components in an arbitrary frame of reference can be defined. Examples are the alternating three-index tensor and the Riemann–Christoffel four-index curvature tensor.

My goal in this book is to present a concise and accessible introduction to vectors and tensors in Cartesian or non-Cartesian, rectilinear or curvilinear coordinates in a way that couples theory and numerical computation.

The notion of uniadic, dyadic, and multiadic bases is emphasized, differential operations on vector and tensor fields inside volumes and over surfaces are derived in terms of the Christoffel symbols and the curvature tensor, and applications in fluid mechanics, membrane theory, and theory of shells are discussed.

Original derivations and novel approaches are presented, including the construction of covariant coordinate fields and the derivation of Green's function of the convection–diffusion equation.

In Chapter 1, the concept of vectors endowed with magnitude and orientation is introduced and the description of a vector in terms of its components in a specified base is discussed. Transformation rules for vector components naturally leads us to the notion of vectors as first-order tensors as opposed to mere one-dimensional numerical arrays. Dyadic bases and the concept of tensors are introduced in a similar way, the description of tensors in terms of their components in a specified base is discussed, and transformation rules are established.

In Chapter 2, vectors and tensor representations in dual biorthogonal bases are discussed, the apparatus of covariant and contravariant bases and associated components is explained, and relevant transformation rules for vectors and tensors are established. The discussion serves as a natural introduction to the subject of curvilinear coordinates where biorthonormal bases are constructed with reference to contravariant and covariant coordinates defined by families of curved lines in space. This natural introduction serves to emphasize that a tensor is a tensor is a tensor: the concept of covariant and contravariant tensors is not appropriate.

In Chapter 3, basic notions and fundamental concepts underlying the structure, construction, and properties of curvilinear coordinates in two dimensions are discussed. In particular, the dual bases discussed in Chapter 2 are reintroduced with reference to contravariant and covariant coordinates and associated base vectors. Following this introduction, finite-difference methods for solving the Laplace and Poisson equations on structured grids are developed and implemented to demonstrate the practical usefulness of the theoretical apparatus.

In Chapter 4, a comprehensive discussion of tensors in non-Cartesian coordinates is presented from the viewpoint of applied mathematics, physics, and engineering. The Christoffel symbols are defined in terms of derivatives of covariant base vectors with respect to contravariant coordinates, and the notion of covariant derivatives of vector and tensor components is discussed. A covariant derivative is a derivative of a vector or tensor component with respect to a contravariant coordinate.

In Chapter 5, vector and tensor calculus on non-Cartesian coordinates discussed, and expressions for the divergence, the curl, the gradient, the Laplacian, and other differential operations are derived using an expedient method that circumvents a great deal of manipulations. Having introduced the necessary framework, applications in mathematical physics are discussed. While stating the contravariant or covariant components of the governing equations in arbitrary curvilinear coordinates is straightforward, subtleties arise in the case of moving coordinates. The notion of convected coordinates is introduced and expressions for Green's functions of the convection–diffusion equation are derived.

In Chapter 6, the apparatus of curvilinear coordinates is specialized to surfaces embedded in space with the introduction of the curvature tensor. Surface calculus is discussed and the Gauss surface divergence theorem is established with applications to force and bending moment equilibria of membranes and thin shells.

A suite of Matlab[1] codes encapsulated in a library named TUNLIB accompany the text. The codes confirm theoretical derivations presented in the book and encode numerical methods for computing solutions of selected partial differential equations. The owner of this book can download TUNLIB freely from the book Internet site: `http://dehesa.freeshell.org/TUN`

[1]Matlab® is a proprietary computing environment for numerical computation and data visualization. Matlab and Simulink are registered trademarks of The MathWorks, Inc. For product information, please contact: The MathWorks, Inc., 3 Apple Hill Drive, Natick, MA 01760-2098, USA, Tel: 508-647-7000, Fax: 508-647-7001, E-mail: info@mathworks.com, Web: www.mathworks.com

This book is suitable for self study and as a text in an upper-level undergraduate or graduate level core or elective course. The theoretical discussion and computational developments assume an upper-level undergraduate or entry-level graduate level knowledge of applied mathematics on readily accessible topics.

C. Pozrikidis

2026

Notation

a	Scalars are set in italic
$\mathbf{A}$	Vectors and matrices are set in bold face
$\mathbf{A}^{\mathrm{T}}$	Matrix transpose
$\mathbf{A}^{-1}$	Matrix inverse
$\cdot$	Vector inner product or matrix product
$\times$	Outer product of two vectors
$\otimes$	Tensor product of two vectors
$\boldsymbol{\nabla}$	Gradient (nabla) operator
∇^2	Laplacian operator
$\nabla^2 f = 0$	Laplace equation
$\nabla^2 f + g = 0$	Poisson equation
$\nabla^2 f + \kappa f = 0$	Helmholtz equation
δ_{ij}	Kronecker's delta
ϵ_{ijk}	Levi–Civita symbol (alternaing tensor)
$\mathbf{g}_i$	covariant base vector
x^i	contravariant coordinate
g_{ij}	covariant metric coefficients
$\mathbf{g}^i$	contravariant base vector
x_i	covariant coordinate
g^{ij}	contravariant metric coefficients
v_α	Cartesian vector component
v_i	contravariant vector component
v^i	covariant vector component

$\mathbf{P}$	tangential projection operator
H	Gaussian curvature
$\mathbf{B}$	curvature tensor
κ_m	mean curvature
$\boldsymbol{\Gamma}_{ij}$	Christoffel symbol
Γ_{ij}^k	Christoffel symbol of the second kind

Chapter 1

Vectors and tensors

Vectors admit rigorous and informal interpretations: physical, conceptual, and as members of mathematical spaces endowed with specific measures, properties, and interaction rules.

In computational practice, vectors are represented by one-dimensional arrays whose elements are the vector components in a chosen base. Tensors are represented by higher-dimensional arrays arranged into matrices whose elements are the tensor components in a dyadic or higher-dimensional base.

Transformation rules for matrix components in different bases and coordinate systems lead us to the notion of tensors. A matrix is qualified as a tensor if the associated matrix components transform according to precise geometrical rules.

1.1 Physical and conceptual vectors

A physical or conceptual vector is a physical or conceptual entity endowed with two attributes: (a) magnitude or length, and (b) direction and orientation in a two-dimensional plane, three-dimensional space, or higher-dimensional hyperspace.

Examples are the position vector defined with respect to a specified origin where an observer resides, a vector indicating the flow of automobile traffic, the flow of information, the motion of a cloud, the translational velocity of a designated center of a rigid body in motion, the angular velocity vector, the wind velocity at a certain location, the force, and many others.

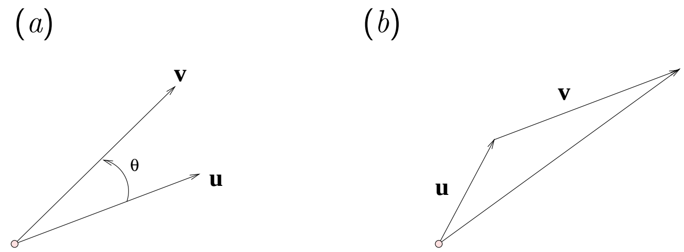

FIGURE 1.1.1 (a) Illustration of vectors endowed with length (magnitude) and orientation in N-dimensional space. Two vectors, $\mathbf{v}$ and $\mathbf{u}$, are inclined at a physical or conceptual angle, θ. (b) To add two vectors, we pipeline them from beginning to end and then connect the first end point to the last end point.

1.1.1 First and second end-points

A conceptual vector can be envisioned as having a starting or first point and an ending or second point. In the case of a free vector, this ordered pair of points can be translated, but not rotated, freely in space as an ordered pair. In the case of a pinned vector, the end points are fixed. The word *vector* describes in Latin the conveyance of the starting point to the ending point.

A vector in a two-dimensional plane or three-dimensional space is drawn as a straight segment with an arrow at the second point away from the first point, as shown in Figure 1.1.1.

1.1.2 Multiplication by a scalar

If we multiply a vector by a positive or negative scalar, we will change its length by a factor that is equal to the magnitude of the scalar, preserve the direction, and either maintain the orientation if the scalar is positive or flip the orientation if the scalar is negative. For example, if we multiply a vector by -1, we will flip the vector into the opposite direction while leaving the magnitude of the vector unchanged.

1.1.3 Addition and subtraction

To add two vectors, we pipeline them from beginning to end in an arbitrary order, as shown in Figure 1.1.1(b). The sum is another vector whose first point is the first point of the first vector and second point is the second point of the second vector. The magnitude and orientation of the new vector are determined by those of the added vectors.

To subtract a vector from another vector, we flip the subtracted vector and then perform addition.

1.1.4 Parallel vectors

If adding one vector to another vectors does not change the direction but may possibly flip the orientation of the first vector, the two vectors are either parallel or anti-parallel.

1.1.5 Relative orientation and inner product

The inner product of a vector, $\mathbf{v}$, with another vector, $\mathbf{u}$, is a scalar denoted by a centered dot $(\cdot)$ and defined as

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}|\,|\mathbf{u}|\,\cos\theta, \tag{1.1.1}$$

where the vertical bars denote the magnitude of the enclosed vector and θ is the physical or conceptual angle between the two vectors varying between and including 0 and π, as shown in Figure 1.1.1(a).

If $\theta = \frac{1}{2}\pi$, the two vectors are mutually orthogonal. If $\theta = 0$, the two vectors are parallel. If $\theta = \pi$, the two vectors are anti-parallel. By the definition of the inner product,

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}. \tag{1.1.2}$$

The inner product is often called a projection.

1.1.6 Vector length or norm

The square of the length or norm of a vector, $\mathbf{v}$, denoted by $|\mathbf{v}|$, is the self-inner product, also called the self-projection,

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}. \tag{1.1.3}$$

This formula arises from $(1.1.1)$ by setting $\mathbf{u} = \mathbf{v}$ and $\theta = 0$. This definition makes an essential association between the length of a vector and the scalar generated by the inner product.

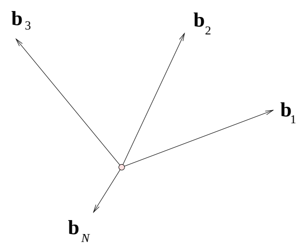

FIGURE 1.2.1 Illustration of a collection of N linearly independent vectors comprising a base.

1.1.7 Outer product

The outer product of an ordered pair of two vectors is a new vector that is perpendicular to the plane or hyperplane hosting the two vectors. The magnitude of the new vector is the area, volume, or hypervolume of the parallelepiped or hyper-parallelepiped formulated by the two vectors in their space. In two or three dimensions, the orientation of the new vector is determined by the right-hand rule.

Exercise

1.1.1 Explain in physical terms why, in order to incline a vector, we may add another orthogonal vector.

1.2 Vector base and vector components

Consider a set of N arbitrary vectors,

$$\mathbf{b}_1, \qquad \mathbf{b}_2, \qquad \ldots, \qquad \mathbf{b}_N, \tag{1.2.1}$$

comprising a vector base, as shown in Figure 1.2.1. This means that no vector in the base can be expressed as a linear combination of the other vectors.

An arbitrary vector, $\mathbf{v}$, can be resolved into a weighted sum of these base vectors,

$$\mathbf{v} = c_1\,\mathbf{b}_1 + \cdots + c_N\mathbf{b}_N = \sum_{i=1}^{N} c_i\,\mathbf{b}_i, \tag{1.2.2}$$

where multiplication by a scalar coefficient, c_i, is followed by addition using the rules of vector manipulation discussed in Section 1.1.

The N scalar coefficients, c_i, are the *components* of the vector $\mathbf{v}$ in the vector base, $\mathbf{b}_i$. In applications, the base vectors $\mathbf{b}_i$ are dimensionless, while the vector components carry physical units, such as mass, time, or length attributed to $\mathbf{v}$. A dimensionless vector can be displayed or graphed in a space of dimensionless axes.

The vector components, c_i, were intentionally denoted by a symbol other than v_i; the latter denote vector components in the universal or laboratory Cartesian system, as discussed in Section 1.2.4.

1.2.1 Einstein summation convention

The Einstein summation convention stipulates that, *if an index appears twice in a product, summation over that index is implied.* If an index appears more then twice, then summation is not implied. A free index appears only once.

Subject to the Einstein summation convention, expansion $(1.2.2)$ is written without the summation symbol as

$$\mathbf{v} = c_i\,\mathbf{b}_i, \tag{1.2.3}$$

where i is summed implicitly from 1 to N. In general, the summation range is not stated explicitly but rather implied.

1.2.2 Multiplication by a scalar

Using expansion (1.2.3), we find that the product of a vector, $\mathbf{v}$, with a scalar, α, is a new vector, $\mathbf{u}$, with components $\alpha\,c_i$,

$$\mathbf{u} \equiv \alpha\,\mathbf{v} = (\alpha\,c_i)\,\mathbf{b}_i, \tag{1.2.4}$$

where summation is implied over the repeated index, i. The distributive property of multiplication has been invoked to write this equation.

1.2.3 Addition

The sum of a vector, $\mathbf{v} = c_i\,\mathbf{b}_i$, and another vector, $\mathbf{u} = d_i\mathbf{b}_i$, is a new vector, $\mathbf{w} = q_i\,\mathbf{b}_i$, with components $q_i = c_i + d_i$, that is,

$$\mathbf{w} = \mathbf{v} + \mathbf{u} = (c_i + d_i)\,\mathbf{b}_i. \tag{1.2.5}$$

To find the sum of two vectors, we merely add their components in a specified vector base.

1.2.4 Subtraction

The difference between a vector, $\mathbf{v} = c_i\,\mathbf{b}_i$, and another vector, $\mathbf{u} = d_i\,\mathbf{b}_i$, is a new vector, $\mathbf{p} = q_i\,\mathbf{b}_i$, where $p_i = c_i - d_i$, that is,

$$\mathbf{p} = \mathbf{v} - \mathbf{u} = (c_i - d_i)\,\mathbf{b}_i. \tag{1.2.6}$$

To find the difference between two vectors, we merely subtract their components in a specified vector base.

1.2.5 Inner product

The inner product of a vector, $\mathbf{v} = c_i\,\mathbf{b}_i$, with another vector $\mathbf{u} = d_j\,\mathbf{b}_j$, is a scalar given by

$$\mathbf{v}\cdot\mathbf{u} = (c_i\mathbf{b}_i)\cdot(d_j\mathbf{b}_j) = c_i\,d_j\,(\mathbf{b}_i\cdot\mathbf{b}_j), \tag{1.2.7}$$

where summation is implied over the repeated indices, i and j, and the untilded and tilded bases can be different or the same.

Expressing the inner products of the base vectors $\mathbf{b}_i$ and $\mathbf{b}_j$ in terms of the angle subtended between these vectors, denoted by $\theta^{(ij)}$ and defined such that

$$\mathbf{b}_i\cdot\mathbf{b}_j = |\mathbf{b}_i|\,|\mathbf{b}_j|\cos\theta^{(ij)}, \tag{1.2.8}$$

we obtain

$$\mathbf{v}\cdot\mathbf{u} = c_i\,d_j\,|\mathbf{b}_i|\,|\mathbf{b}_j|\,\cos\theta^{(ij)}, \tag{1.2.9}$$

where summation is implied over the repeated indices, i and j.

1.2.6 Computation of vector components

To compute the components of a vector $\mathbf{v}$, in a specified base, c_m, we may take the inner product of expansion (1.2.3) with $\mathbf{b}_m$, where m is a free index. The result is a system of N linear equations for the vector components,

$$(\mathbf{b}_m \cdot \mathbf{b}_i)\, c_i = \mathbf{v} \cdot \mathbf{b}_m, \tag{1.2.10}$$

where summation is implied over the repeated index, i. Explicitly, the linear system takes the form

$$\begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{b}_1 \cdot \mathbf{b}_N \\ \vdots & \vdots & \\ \mathbf{b}_N \cdot \mathbf{b}_1 & \cdots & \mathbf{b}_N \cdot \mathbf{b}_N \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \mathbf{v} \cdot \mathbf{b}_1 \\ \vdots \\ \mathbf{v} \cdot \mathbf{b}_N \end{bmatrix}. \tag{1.2.11}$$

This linear system can be solved by standard analytical or numerical methods, such as Cramer's rule, Gauss elimination, or LU decomposition. Note that, if all base vectors are mutually orthogonal, only the diagonal elements of the matrix on the left-hand side survive. Consequently, the computation of the vector components is considerably simplified.

Expressing the inner products in terms of (a) the relative angles between the vectors $\mathbf{b}_m$ and $\mathbf{b}_i$, denoted by $\theta^{(mi)}$ and defined such that

$$\mathbf{b}_m \cdot \mathbf{b}_i = |\mathbf{b}_m|\, |\mathbf{b}_i| \cos\theta^{(mi)}, \tag{1.2.12}$$

and (b) the relative angle between $\mathbf{v}$ and $\mathbf{b}_m$, denoted by $\theta^{(m)}$ and defined such that

$$\mathbf{v} \cdot \mathbf{b}_m = |\mathbf{v}|\, |\mathbf{b}_m| \cos\theta^{(m)}, \tag{1.2.13}$$

we obtain

$$|\mathbf{b}_i| \cos\theta^{(mi)}\, c_i = |\mathbf{v}|\, \cos\theta^{(m)} \tag{1.2.14}$$

for $m = 1, \ldots, N$, where summation is implied over the repeated index, i. The linear system (1.2.11) takes the form

$$\begin{bmatrix} |\mathbf{b}_1| \cos\theta^{(11)} & \cdots & |\mathbf{b}_1| \, \cos\theta^{(1N)} \\ \vdots & \vdots & \\ |\mathbf{b}_N| \, \cos\theta^{(N1)} & \cdots & |\mathbf{b}_N| \, \cos\theta^{(NN)} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = |\mathbf{v}| \begin{bmatrix} \cos\theta^{(1)} \\ \vdots \\ \cos\theta^{(N)} \end{bmatrix}, \tag{1.2.15}$$

where $\cos\theta^{(ii)} = 1$. The solution can be found by standard numerical methods.

1.2.7 Component array

The vector components, c_i, can be encapsulated in an array, $\mathbf{c}$, whose elements depend on the chosen vector base, $\mathbf{b}_i$,

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}. \tag{1.2.16}$$

The vector $\mathbf{v}$ can be reconstructed from this array and associated base vectors using the rules of scalar–vector multiplication and vector addition.

If the component arrays of two vectors are equal, the two vectors are the same. Conversely, if two vectors are identical, their component arrays are equal.

1.2.8 Object described by components

An ellipse is an object described either in terms of its two axes or in terms of one axis and the aspect ratio. In the extensible markup language (XML) notation, an ellipse is described by the following statement involving two attributes:

```
<ellipse first_axis="0.1" second_axis="0.2"/>
```

An alternative description is:

```
<shape type="ellipse">
  <first_axis>0.1</first_axis>
  <second_axis>0.2</second_axis>
</shape>
```

where the first and second axes are attributes assigned numerical values in some agreed units.

A certain vector, $\mathbf{v}$, may also be regarded as an object, and the associated component array pertaining to a certain base, $\mathbf{c}$, may be regarded as a host of encapsulated attributes representing the fingerprint of the object in the specified base. The base itself implements an observational framework.

The essential features of the object should be independent of the observational framework. The interpretation of an entity as an object independent of its fingerprint or projection in an observational framework underlies the notion of a tensor.

1.2.9 Orthogonal base

In the event that the base vectors $\mathbf{b}_i$ are mutually orthogonal, denoted by $\mathbf{o}_i$, the relative angles are $\theta^{(ij)} = 90°$ for $i \neq j$. Consequently, $\cos\theta^{(ij)} = 0$ and

$$\mathbf{o}_i \cdot \mathbf{o}_j = 0 \tag{1.2.17}$$

for $i \neq j$. The linear expansion of an arbitrary vector takes the form

$$\mathbf{v} = c_i \, \mathbf{o}_i. \tag{1.2.18}$$

Equation (1.2.14) provides us with an explicit expression for the vector components,

$$c_i = \frac{|\mathbf{v}|}{|\mathbf{o}_i|} \cos\theta_i, \tag{1.2.19}$$

where θ_i are direction cosines of the angles subtended between $\mathbf{v}$ and $\mathbf{o}_j$.

1.2.10 Cartesian base

When the length of each orthogonal base vector is unity, $|\mathbf{o}_i| = 1$, the vector base is Cartesian. Cartesian base vectors are denoted as $\mathbf{e}_i$. By definition,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.2.20}$$

where δ_{ij} is Kronecker's delta representing the identity matrix: $\delta_{ij} = 1$ if $i = j$, or 0 otherwise.

The linear expansion of an arbitrary vector, $\mathbf{v}$, takes the form

$$\mathbf{v} = c_i \, \mathbf{e}_i, \tag{1.2.21}$$

where summation is implied over the repeated index, i. Equation (1.2.19) provides us with the Cartesian array components

$$c_i = |\mathbf{v}| \, \cos\theta_i, \tag{1.2.22}$$

where θ_i are direction cosines between $\mathbf{v}$ and $\mathbf{e}_i$ in N-dimensional space. When arranged in an array, the components c_i provide us with a Cartesian vector, as discussed in Section 1.4.

Exercise

1.2.1 Factorize the matrix on the right-hand side of (1.2.15) into a diagonal and a symmetric matrix.

1.3 Change of base

We may introduce an alternative set of base vectors indicated by a tilde, $\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_N$, and expand an arbitrary N-dimensional vector, $\mathbf{v}$, as

$$\mathbf{v} = \widetilde{c}_i \, \widetilde{\mathbf{b}}_i, \tag{1.3.1}$$

where $\widetilde{c}_i$ are the components of $\mathbf{v}$ in the tilded base encapsulated in the array

$$\widetilde{\mathbf{c}} = \begin{bmatrix} \widetilde{c}_1 \\ \vdots \\ \widetilde{c}_N \end{bmatrix}. \tag{1.3.2}$$

The vector $\mathbf{v}$ can be reconstructed from this alternative array and associated base vectors.

1.3.1 Relation between base vectors

We expect that the array $\mathbf{c}$ will be related to the alternative array $\widetilde{\mathbf{c}}$, in that one can be deduced from the other by a suitable transformation. To confirm this expectation, we express each base vector of the tilded base in terms of base vectors in the untilded base using a linear transformation,

$$\widetilde{\mathbf{b}}_i = H_{ij}\,\mathbf{b}_j, \tag{1.3.3}$$

where $\mathbf{H}$ is a transformation matrix and summation is implied over the repeated index, j. The matrix $\mathbf{H}$ does not necessarily have any specific properties, that is, it not necessarily orthogonal or symmetric.

Multiplying the linear expansion (1.3.3) by the inverse element H^{-1}_{mi} and summing over i, we obtain

$$H^{-1}_{mi}\,\widetilde{\mathbf{b}}_i = H^{-1}_{mi}\,H_{ij}\,\mathbf{b}_j, \tag{1.3.4}$$

where m is a free index and the superscript -1 denotes the matrix inverse. Now invoking the definition of the matrix inverse, we set

$$H^{-1}_{mi}\,H_{ij} = \delta_{mj}, \tag{1.3.5}$$

where δ_{mj} is Kronecker's delta representing the identity matrix: $\delta_{mj} = 1$ if $m = j$, or 0 otherwise. Setting $\delta_{mj}\,\mathbf{b}_j = \mathbf{b}_m$ and rearranging, we obtain

$$\mathbf{b}_m = H^{-1}_{mi}\,\widetilde{\mathbf{b}}_i, \tag{1.3.6}$$

which is the companion of (1.3.3). If the matrix $\mathbf{H}$ happens to be orthogonal, the inverse is equal to the transpose and $H^{-1}_{mi} = H_{im}$.

1.3.2 Relation between vector component arrays

Substituting (1.3.6) into (1.2.3), we obtain

$$\mathbf{v} = c_i\,H^{-1}_{ij}\,\widetilde{\mathbf{b}}_j, \tag{1.3.7}$$

where summation is implied over the repeated indices, i and j. Comparing this expansion with (1.3.1), we conclude that

$$\widetilde{c}_j = c_i\, H_{ij}^{-1} = H_{ji}^{-\mathrm{T}} c_i \tag{1.3.8}$$

or

$$\widetilde{\mathbf{c}} = \mathbf{H}^{-\mathrm{T}} \cdot \mathbf{c}, \tag{1.3.9}$$

where the superscript $-\mathrm{T}$ denotes the inverse of the transpose. To invert this relation, we multiply both sides with the transpose of the matrix $\mathbf{H}$ and obtain

$$c_i = H_{ij}^{\mathrm{T}}\, \widetilde{c}_j = H_{ji}\, \widetilde{c}_j, \tag{1.3.10}$$

or

$$\mathbf{c} = \mathbf{H}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}. \tag{1.3.11}$$

Note the partial similarity of the pair of (1.3.9) and (1.3.11) with the pair of (1.3.3) and (1.3.6).

1.3.3 First-order tensors

If the components of a vector conform with the transformation rule (1.3.9) and its inverse rule (1.3.11), summarized below,

$$\widetilde{\mathbf{c}} = \mathbf{H}^{-\mathrm{T}} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{H}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}, \tag{1.3.12}$$

then the vector is accepted as a *first-order* tensor. If they do not, the vector is regarded as a mere numerical array all too familiar to computed programmers.

Any physical vector employed in the physical sciences and engineering is a first-order tensor. Examples are the position, the velocity, the angular velocity, the force, and torque, and other similar vectors endowed with magnitude and direction.

1.3.4 Note on notation

In the literature, the transpose of the transformation matrix $\mathbf{H}$ is sometimes employed in equation (1.3.3), by writing

$$\widetilde{\mathbf{b}}_i = L_{ji}\, \mathbf{b}_j, \tag{1.3.13}$$

where $\mathbf{L} = \mathbf{H}^{\mathrm{T}}$ and the superscript T denotes the transpose. Correspondingly, the vector components are related by

$$\widetilde{\mathbf{c}} = \mathbf{L}^{-1} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{L} \cdot \widetilde{\mathbf{c}}, \tag{1.3.14}$$

Care should be taken so that misinterpretations and errors do not arise due to the juxtaposition of the indices when referring to other sources.

1.3.5 Inner product in terms of vector components

The inner product of a vector, $\mathbf{v} = c_i\, \mathbf{b}_i$, with another vector, $\mathbf{u} = \widetilde{d}_j\, \widetilde{\mathbf{b}}_j$, is a scalar given by

$$\mathbf{v} \cdot \mathbf{u} = (c_i \mathbf{b}_i) \cdot (\widetilde{d}_j \widetilde{\mathbf{b}}_j) = c_i\, \widetilde{d}_j\, (\mathbf{b}_i \cdot \widetilde{\mathbf{b}}_j), \tag{1.3.15}$$

where summation is implied over the repeated indices, i and j, and the untilded and tilded bases can be different or the same. Expressing the inner products of the base vectors $\mathbf{b}_i$ and $\widetilde{\mathbf{b}}_j$ in terms of the angles subtended between these vectors, denoted by $\varphi^{(ij)}$ and defined such that

$$\mathbf{b}_i \cdot \widetilde{\mathbf{b}}_j = |\mathbf{b}_i|\, |\widetilde{\mathbf{b}}_j|\, \cos \varphi^{(ij)}, \tag{1.3.16}$$

we obtain

$$\mathbf{v} \cdot \mathbf{u} = c_i\, \widetilde{d}_j\, |\mathbf{b}_i|\, |\widetilde{\mathbf{b}}_j|\, \cos \varphi^{(ij)}, \tag{1.3.17}$$

where summation is implied over the repeated indices, i and j.

Exercise

1.3.1 Derive (1.3.11).

1.4 Cartesian vectors

The discussion of physical and conceptual vectors in the first three sections of this chapter hinges on four notions: (a) the notion of a vector as an object, (b) the notion of vector multiplication by a scalar, (c) the notion of vector addition, and (d) the notion vector inner product defined in terms of the magnitude of the participating vectors and relative inclination angle.

An extended algebraic framework that allows us to conduct further analysis and perform numerical computations can be established by introducing the notion of the universal or laboratory Cartesian base.

1.4.1 Universal Cartesian base

Each base vector in the universal Cartesian base is associated with a dimensionless N-dimensional numerical array denoted by $\boldsymbol{\epsilon}_i$ for $i = 1, \ldots, N$. By definition, all entries of the universal base arrays $\boldsymbol{\epsilon}_i$ are zero, except for the ith entry that is equal to 1. In three dimensions,

$$\boldsymbol{\epsilon}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad (1.4.1)$$

In the literature, these base arrays are denoted by a variety of symbols, including $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$.

Next, we consider a physical or conceptual vector, $\mathbf{v}$, and introduce the expansion

$$\mathbf{v} = v_i \, \boldsymbol{\epsilon}_i, \qquad (1.4.2)$$

where summation is implied over the repeated index, i. Subject to the preceding definitions, the physical vector, $\mathbf{v}$, is now identified with a numerical array, $\mathbf{v}$, that is identical to the component array in the universal Cartesian base,

$$\mathbf{v} = v_i \, \boldsymbol{\epsilon}_i = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}. \qquad (1.4.3)$$

The correspondence between a physical vector and a Cartesian array is a fundamental equivalence concept.

1.4.2 Physical units

The physical units of a vector, $\mathbf{v}$, such as length, are conveyed by the Cartesian components, v_i. All components are assumed to have the same units, that is, inhomogeneous arrays are not allowed. Cartesian vectors whose components have different dimensions cannot be handled by the standard framework.

1.4.3 Arbitrary Cartesian base

The arrays of an arbitrary Cartesian base satisfy the orthonormality condition

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.4.4}$$

where δ_{ij} is Kronecker's delta.

For example, in two dimensions, we may have

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{1.4.5}$$

The length of each one of these vectors is unity, and the two vectors are orthogonal.

This example underscores that a distinction should be made between the universal Cartesian base, $\boldsymbol{\epsilon}_i$, and an arbitrary Cartesian base, $\mathbf{e}_i$. A Cartesian base described by a set of Cartesian arrays, $\mathbf{e}_i$, is not necessarily the universal base, $\boldsymbol{\epsilon}_i$.

1.4.4 Matrix of Cartesian base vectors

It is useful to introduce a matrix $\mathbf{E}$ hosting in its columns the Cartesian vectors of a certain base,

$$\mathbf{E} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \cdots & \mathbf{e}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \tag{1.4.6}$$

defined such that E_{ij} is the ith component of $\mathbf{e}_j$ in the universal Cartesian base. By definition,

$$\mathbf{e}_j = \begin{bmatrix} E_{1j} \\ \vdots \\ E_{Nj} \end{bmatrix} = \boldsymbol{\epsilon}_i \, E_{ij}, \tag{1.4.7}$$

where summation is implied over the repeated index, i. If the Cartesian base $\mathbf{e}_i$ is the universal base, $\boldsymbol{\epsilon}_i$, then the matrix $\mathbf{E}$ is the identity matrix.

Since the projection of any column onto any other column is zero and the projection of any column onto itself is unity,

$$\mathbf{E}^{\mathrm{T}} \cdot \mathbf{E} = \mathbf{I}, \tag{1.4.8}$$

where the superscript T denotes the matrix transpose and $\mathbf{I}$ is the identity matrix. This relation demonstrates that the matrix $\mathbf{E}$ is orthogonal, satisfying

$$\mathbf{E}^{\mathrm{T}} = \mathbf{E}^{-1}, \qquad \mathbf{E}^{-\mathrm{T}} = \mathbf{E}, \tag{1.4.9}$$

where the superscript -1 denotes the matrix inverse and the superscript $-\mathrm{T}$ denotes the inverse of the transpose.

For example, with reference to the base defined in (1.4.5),

$$\mathbf{E} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{E}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{1.4.10}$$

We may readily verify that the property $\mathbf{E} \cdot \mathbf{E}^{\mathrm{T}} = \mathbf{I}$ is satisfied.

1.4.5 Vector components

An arbitrary vector, $\mathbf{v}$, can be expanded in an arbitrary Cartesian base as

$$\mathbf{v} = c_j \, \mathbf{e}_j, \tag{1.4.11}$$

where c_j are the vector components. If $\mathbf{e}_j$ is the universal base, $\mathbf{e}_j = \boldsymbol{\epsilon}_j$, and only then, $c_j = v_j$ according to (1.4.3). Substituting into this expansion expression (1.4.7), we find that

$$\mathbf{v} = c_j \, \mathbf{e}_j = E_{ij} \, c_j \, \boldsymbol{\epsilon}_i, \tag{1.4.12}$$

where summation is implied over the repeated indices, i and j. This expression shows that

$$\mathbf{v} = \mathbf{E} \cdot \mathbf{c}, \tag{1.4.13}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \tag{1.4.14}$$

is the component array. Conversely,

$$\mathbf{c} = \mathbf{E}^{\mathrm{T}} \cdot \mathbf{v}, \tag{1.4.15}$$

where the superscript T denotes the matrix transpose. The component array can be deduced by a mere matrix–vector multiplication.

For example, with reference to the base defined in (1.4.5), if $c_1 = 1$ and $c_2 = 1$, then

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{1.4.16}$$

In practice, given a vector, $\mathbf{v}$, and a base, $\mathbf{e}_i$, we will be interested in computing the vector components, c_i.

1.4.6 Frame independence

A physical vector described by a Cartesian array, $\mathbf{v}$, will have different component vectors, $\mathbf{c}$, in different Cartesian bases. If we change the Cartesian base, $\mathbf{v}$ will remain the same, but the entries of the component array $\mathbf{c}$ will be modified. A physical vector is a physical vector no matter how it is described in terms of a chosen base. We say that a physical vector is invariant or frame-independent. This property is the cornerstone of a tensor.

1.4.7 Not every array represents a Cartesian vector

Every physical vector can be regarded as a Cartesian array, and *vice versa*. However, not every array encountered in science, engineering, and elsewhere represents a physical or conceptual vector. Vector component transformation rules must be obeyed, as discussed in Section 1.8.

Exercise

1.4.1 Confirm equation (1.4.13).

1.5 Vector inner and outer products

A pair of vectors can be multiplied in three ways: inner (dot) product ($\cdot$), outer (cross) product ($\times$), tensor (dyadic or Cartesian) product ($\otimes$). The inner product is a scalar, the outer product is a vector, and the tensor product is a matrix. The inner and outer products are discussed in this section, while the tensor product is discussed in Section 1.6.

1.5.1 Inner product

The inner product of a vector, $\mathbf{v} = c_i \mathbf{e}_i$, with another vector, $\mathbf{u} = d_j \mathbf{e}_j$, is a scalar denoted by a centered dot,

$$\mathbf{v} \cdot \mathbf{u} = (c_i \, \mathbf{e}_i) \cdot (d_j \, \mathbf{e}_j). \tag{1.5.1}$$

Expanding the product, we write

$$\mathbf{v} \cdot \mathbf{u} = c_i \, d_j \, (\mathbf{e}_i \cdot \mathbf{e}_j) = c_i \, d_j \, \delta_{ij} = c_i \, d_i, \tag{1.5.2}$$

where δ_{ij} is Kronecker's delta representing the identity matrix. Consequently,

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{u}| \, |\mathbf{v}| \, \cos\theta = c_i \, d_i, \tag{1.5.3}$$

where summation is implied over the repeated index, i, and θ is the relative inclination angle between the two vectors. We recall that, in the universal frame, $c_i = v_i$ and $d_i = u_i$.

1.5.2 Magnitude

The magnitude of a Cartesian vector, $\mathbf{v}$, is

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{c_i c_i} = |\mathbf{c}|, \tag{1.5.4}$$

where summation is implied over the repeated index, i. We expect and will confirm that the magnitude $|\mathbf{v}|$ is insensitive to the choice of Cartesian base, that is, it will depend only on the Euclidean norm of the component array, $|\mathbf{c}|$.

1.5.3 Relative inclination angle

Referring to (1.5.3), we find that the relative inclination angle between two vectors, $\mathbf{v} = c_i\mathbf{e}_i$ and $\mathbf{u} = d_i\mathbf{e}_i$, is given by

$$\theta = \arctan\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|} = \arctan\frac{c_i\,d_i}{|\mathbf{c}|\,|\mathbf{d}|} \tag{1.5.5}$$

in the range $[0, \pi]$, where summation is implied over the repeated index, i. This expression allows us to compute the relative inclination angle, θ, in terms of the components of the two vectors involved.

1.5.4 Physical angles

The relative angle θ is physical in two or three dimensions, and conceptual in higher dimensions. To show this, we recall that $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$ and $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$, and then write

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,\mathbf{v}\cdot\mathbf{u}. \tag{1.5.6}$$

By the law of cosines,

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{v}||\mathbf{u}|\cos\theta. \tag{1.5.7}$$

Comparing these two equations, we find that

$$\mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta, \tag{1.5.8}$$

which proves the assertion.

1.5.5 Cross product

The cross or outer product of an ordered pair of vectors, $\mathbf{v} = c_i \mathbf{e}_i$ and $\mathbf{u} = d_j \mathbf{e}_j$, is a new vector,

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = (c_i\, \mathbf{e}_i) \times (c_j\, \mathbf{e}_j) = c_i\, d_j\, \mathbf{e}_i \times \mathbf{e}_j, \tag{1.5.9}$$

where $\mathbf{e}_i \times \mathbf{e}_j$ is a set of Cartesian vectors best defined in terms of the Levi–Civita symbol discussed next.

1.5.6 Levi–Civita symbol

In three dimensions, $N = 3$ the Levi–Civita symbol, ϵ_{ijk}, is defined such that $\epsilon_{ijk} = 1$ for a cyclic permutation of indices, -1 for an anti-cyclic permutation, and 0 otherwise. For example, $\epsilon_{312} = 1$ and $\epsilon_{221} = 0$. Formally, we define

$$\epsilon_{ijk} = \tfrac{1}{2}\,(i-j)(j-k)(k-i) \tag{1.5.10}$$

for $i, j, k = 1, 2, 3$. For example,

$$\epsilon_{123} = \tfrac{1}{2}\,(1-2)(2-3)(3-1) = 1. \tag{1.5.11}$$

It can be shown that

$$\epsilon_{ijk}\,\epsilon_{lmn} = \det \begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix}, \tag{1.5.12}$$

where i, j, k, l, m, n is a collection of six free indices.

Three useful properties of the Levi–Civita symbol originating from (1.5.12) are

$$\begin{aligned} &\epsilon_{ijk}\,\epsilon_{ijk} = 6, \qquad \epsilon_{ijk}\,\epsilon_{ijl} = 2\,\delta_{kl}, \\ &\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}, \end{aligned} \tag{1.5.13}$$

where summation is implied over indices that appear twice. Note that the first identity involves two free indices, while the second identity involves four free indices.

1.5.7 Outer product of two Cartesian base vectors

The outer product of a pair of Cartesian base vectors is given by

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\, \mathbf{e}_k, \tag{1.5.14}$$

where summation is implied over the repeated index, k. Conversely,

$$\epsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k. \tag{1.5.15}$$

The right-hand side is the mixed triple product of the Cartesian base vectors.

In Section 1.16, we will see that the Levi–Civita symbol encapsulates the Cartesian components of the alternating tensor.

1.5.8 Outer product in terms of the Levi–Civita symbol,

Using (1.5.14), we find that the outer or cross product of an ordered pair of vectors, $\mathbf{v} = c_i \mathbf{e}_i$ and $\mathbf{u} = d_j \mathbf{e}_j$, is given by

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = c_i\, d_j\, \epsilon_{ijk}\, \mathbf{e}_k = \epsilon_{kij}\, c_i\, d_j \mathbf{e}_k. \tag{1.5.16}$$

The last expression reveals that the components of the outer product of two three-dimensional vector, $\mathbf{w} = \mathbf{c} \times \mathbf{d}$, are given by

$$w_k = \epsilon_{kij}\, c_i\, d_j = -\epsilon_{kji}\, c_i\, d_j, \tag{1.5.17}$$

where $\mathbf{w} = q_i \mathbf{e}_i$. We see that terms can be freely transposed in a scalar product.

Exercise

1.5.1 Prove (1.5.12).

1.6 Tensor product of two vectors

The tensor product of two vectors, $\mathbf{v} = c_i \mathbf{e}_i$ and $\mathbf{u} = d_i \mathbf{e}_i$, is a two-index Cartesian tensor,

$$\mathbf{T} \equiv \mathbf{v} \otimes \mathbf{u} = (c_i\, \mathbf{e}_i) \otimes (d_j\, \mathbf{e}_j) = c_i d_j\, \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.6.1}$$

The collection of matrices

$$\mathbf{E}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j \tag{1.6.2}$$

defines an orthonormal dyadic base. By definition, the $k\ell$ element of $\mathbf{E}_{ij}$ is

$$[\mathbf{E}_{ij}]_{k\ell} \equiv [\mathbf{e}_i]_k \times [\mathbf{e}_j]_\ell, \tag{1.6.3}$$

where $\times$ now denotes regular number multiplication, $[\mathbf{e}_i]_k$ is the kth element of $\mathbf{e}_i$, and $[\mathbf{e}_j]_\ell$ is the ℓth element of $\mathbf{e}_j$. Subject to these definitions,

$$\mathbf{T} \equiv \mathbf{v} \otimes \mathbf{u} = c_i d_j \, \mathbf{E}_{ij}, \tag{1.6.4}$$

where summation is implied over the repeated indices, i and j.

1.6.1 Trace

By the definition of the tensor product,

$$\mathrm{trace}(\mathbf{e}_i \otimes \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.6.5}$$

and therefore

$$\mathrm{trace}(\mathbf{v} \otimes \mathbf{u}) = \mathbf{v} \cdot \mathbf{u} = c_i \, d_i, \tag{1.6.6}$$

where summation is implied over the repeated index, i.

1.6.2 Universal Cartesian base

Referring to the universal Cartesian base, we set $\mathbf{e}_i = \boldsymbol{\epsilon}_i$ and find that all elements of the matrix $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j$ are zero, except for the ij element that is equal to unity. Consequently, the elements of $\mathbf{T}$ are

$$T_{ij} = v_i u_j, \tag{1.6.7}$$

so that

$$\mathbf{T} \equiv \mathbf{v} \otimes \mathbf{u} = v_i u_j \, \boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j. \tag{1.6.8}$$

The matrix $\mathbf{T}$ is symmetric only if $\mathbf{u}$ and $\mathbf{v}$ are the same.

1.6.3 Dyadic or Cartesian product

The tensor product is also known as the *dyadic* or *Cartesian* product. The term *Cartesian* is used in probability theory with regard to the joint probability distribution. The terms *dyadic* and *tensor* are used in mathematics and mechanics. The significance of the terminology *tensor* will become clear in the following discussion.

1.6.4 Components of the tensor product

We may write

$$\mathbf{T} \equiv \mathbf{v} \otimes \mathbf{u} = C_{ij}\, \mathbf{E}_{ij}, \tag{1.6.9}$$

where summation is implied over the repeated indices i and j, and

$$C_{ij} = c_i\, d_j. \tag{1.6.10}$$

The components, C_{ij}, can be accommodated in an $N \times N$ matrix denoted by $\mathbf{C}$, which is generally different than the matrix $\mathbf{T}$ whose elements are given in (1.6.7). The numbers encapsulated in the component matrix $\mathbf{C}$ depend on the chosen dyadic base defined by the base vectors, $\mathbf{e}_i$. Only in the universal Cartesian base, $\boldsymbol{\epsilon}_i$, the matrix $\mathbf{C}$ is the same as $\mathbf{T}$.

1.6.5 Multiplication properties of the tensor product

For any three vectors, $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$,

$$(\mathbf{v} \otimes \mathbf{u}) \cdot \mathbf{w} = \mathbf{v}\,(\mathbf{u} \cdot \mathbf{w}) \tag{1.6.11}$$

and

$$\mathbf{w} \cdot (\mathbf{v} \otimes \mathbf{u}) = \mathbf{u}\,(\mathbf{w} \cdot \mathbf{v}), \tag{1.6.12}$$

where $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$ are scalar inner products. The properties can be proven readily working in index notation.

The double-dot product of two tensor products is a scalar defined as

$$(\mathbf{v} \otimes \mathbf{u}) : (\mathbf{w} \otimes \mathbf{q}) = (\mathbf{v} \cdot \mathbf{w})(\mathbf{u} \cdot \mathbf{q}). \tag{1.6.13}$$

The last expression is the product of two scalars.

For any three vectors, $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$,

$$(\mathbf{v} \otimes \mathbf{u}) \times \mathbf{w} = \mathbf{v} \otimes (\mathbf{u} \times \mathbf{w}) \tag{1.6.14}$$

and

$$\mathbf{w} \times (\mathbf{v} \otimes \mathbf{u}) = (\mathbf{w} \times \mathbf{v}) \otimes \mathbf{u}, \tag{1.6.15}$$

where $\times$ denotes the cross product. The properties can be proven readily working in index notation.

1.6.6 A singular matrix

The matrix

$$\mathbf{A} = \mathbf{I} - \mathbf{v} \otimes \mathbf{u} \tag{1.6.16}$$

with elements $A_{ij} = \delta_{ij} - v_i u_j$ is singular, subject to the restriction that $\mathbf{v} \cdot \mathbf{u} = 1$. The reason is that at least one eigenvalue is zero with corresponding eigenvector $\mathbf{v}$, that is, $\mathbf{A} \cdot \mathbf{v} = \mathbf{0}$. The associated eigenvector of the matrix transport (left eigenvector) is $\mathbf{u}$, that is, $\mathbf{A} \cdot \mathbf{u} = \mathbf{0}$.

1.6.7 Matrix representation

A matrix can be regarded as a collection of N vectors, $\mathbf{v}_i$, represented by their components arranged down the matrix columns,

$$\mathbf{A} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}. \tag{1.6.17}$$

Using the properties of the tensor product, we find that

$$\mathbf{A} = \mathbf{v}_k \otimes \boldsymbol{\epsilon}_k, \tag{1.6.18}$$

where $\boldsymbol{\epsilon}_k$ is the universal Cartesian base and summation is implied over the repeated index, k.

Exercise

1.6.1 Compute the component matrix of the tensor product of two vectors, $\mathbf{v} = [1, 2, 3]$ and $\mathbf{u} = [3, 2, 1]$ in the universal Cartesian base.

1.7 Position and coordinates

To locate a point in three-dimensional space, we introduce the position vector, $\mathbf{x}$ connecting the origin of the universal Cartesian system to the point. For convenience, the three elements of $\mathbf{x}$ are denoted by $x_1 = x$, $x_2 = y$, $x_3 = z$, so that

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\,\boldsymbol{\epsilon}_x + y\,\boldsymbol{\epsilon}_y + z\,\boldsymbol{\epsilon}_z. \tag{1.7.1}$$

The triplet $(x_1, x_2, x_3) = (x, y, z)$ constitutes a set of coordinates in the universal Cartesian base.

1.7.1 Arbitrary Cartesian base

With reference to an arbitrary Cartesian system with base vectors $\mathbf{e}_i$, the position is

$$\mathbf{x} = \mathbf{x}_0 + X_1\,\mathbf{e}_1 + X_2\,\mathbf{e}_2 + X_3\,\mathbf{e}_3 = \mathbf{x}_0 + X_i\,\mathbf{e}_i, \tag{1.7.2}$$

where $\mathbf{x}_0$ is the position of the origin of the Cartesian system with respect to that of the universal system. The base vectors are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial X_i}. \tag{1.7.3}$$

Generalized coordinates can be defined in another physical or abstract space.

1.7.2 Differential displacement

A differential displacement is resolved as

$$\mathrm{d}\mathbf{x} = \mathrm{d}x\,\boldsymbol{\epsilon}_x + \mathrm{d}y\,\boldsymbol{\epsilon}_y + \mathrm{d}z\,\boldsymbol{\epsilon}_z = \mathrm{d}X_i\,\mathbf{e}_i. \tag{1.7.4}$$

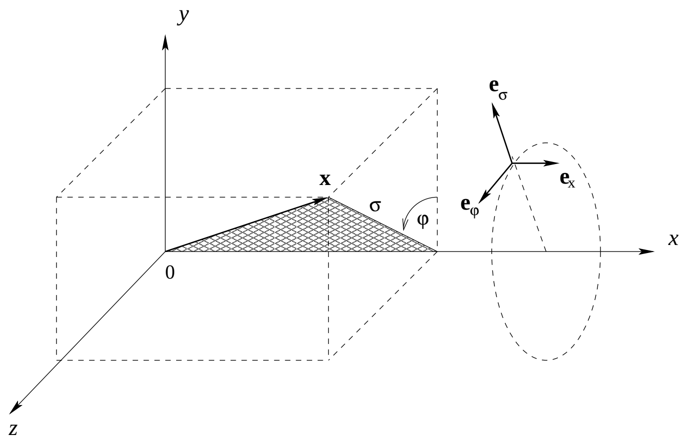

FIGURE 1.7.1 Illustration of cylindrical polar coordinates, (x, σ, φ), defined with respect to Cartesian coordinates, (x, y, z), where σ is the distance from the x axis and φ is the azimuthal angle around the x axis.

The square of the magnitude of the differential displacement is the fundamental form of space, given by

$$\mathrm{d}s^2 \equiv \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = \mathrm{d}X_i \, \mathrm{d}X_i, \qquad (1.7.5)$$

where summation is implied over the repeated index, i. The associated metric coefficients are 1, 1, 1.

1.7.3 Cylindrical polar coordinates

An arbitrary point in space can be identified by the cylindrical polar coordinates, (x, σ, φ), where σ is the distance from the x axis and φ is the azimuthal angle, as illustrated in Figure 1.7.1. The associated base unit vectors are

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_\sigma = \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix}, \qquad \mathbf{e}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}. \quad (1.7.6)$$

Note that the unit vectors depend of φ.

The position vector is

$$\mathbf{x} = x\,\mathbf{e}_x + \sigma\,\mathbf{e}_\sigma \tag{1.7.7}$$

and the differential displacement is

$$\mathrm{d}\mathbf{x} = \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}\sigma\,\mathbf{e}_\sigma + \sigma\,\mathrm{d}\mathbf{e}_\sigma. \tag{1.7.8}$$

Using expressions (1.7.6), we find that $\mathrm{d}\mathbf{e}_\sigma = \mathbf{e}_\varphi\,\mathrm{d}\varphi$, yielding

$$\mathrm{d}\mathbf{x} = \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}\sigma\,\mathbf{e}_\sigma + \sigma\,\mathrm{d}\varphi\,\mathrm{d}\mathbf{e}_\varphi. \tag{1.7.9}$$

The fundamental form of space takes the form

$$\mathrm{d}^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = (\mathrm{d}x)^2 + (\mathrm{d}\sigma)^2 + \sigma^2\,(\mathrm{d}\varphi)^2. \tag{1.7.10}$$

The associated metric coefficients are 1, 1, σ^2.

An arbitrary vector can be resolved as

$$\mathbf{v} = v_x\,\mathbf{e}_x + v_\sigma\,\mathbf{e}_\sigma + v_\varphi\,\mathbf{e}_\varphi, \tag{1.7.11}$$

where v_x, v_σ, and v_φ are the cylindrical polar components of $\mathbf{v}$.

The tensor product of two vectors $\mathbf{v}$ and $\mathbf{u}$ is a matrix,

$$\mathbf{T} = \mathbf{v}\otimes\mathbf{u} = v_\alpha u_\beta\,\mathbf{e}_\alpha\otimes\mathbf{e}_\beta, \tag{1.7.12}$$

where summation of the Greek indices is implied in the range x, σ, φ. For example,

$$\mathbf{e}_x\otimes\mathbf{e}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.7.13}$$

and

$$\mathbf{e}_\varphi\otimes\mathbf{e}_\varphi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin^2\varphi & -\sin\varphi\cos\varphi \\ 0 & -\sin\varphi\cos\varphi & \cos^2\varphi \end{bmatrix}. \tag{1.7.14}$$

Seven similar tensor products of the unit vectors generate similar matrices.

1.7.4 Spherical polar coordinates

An arbitrary point in space can be identified by the spherical polar coordinates, (r, θ, φ), where r is the distance from the origin, θ is the meridional angle, and φ is the azimuthal angle, as illustrated in Figure 1.7.2. The associated base unit vectors are

$$\mathbf{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \end{bmatrix}, \quad \mathbf{e}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}, \quad \mathbf{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix}. \tag{1.7.15}$$

Note that the unit vectors depend of θ and φ.

The position vector is described in terms of the radial base vector alone,

$$\mathbf{x} = r\,\mathbf{e}_r. \tag{1.7.16}$$

The differential displacement is

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\mathbf{e}_r. \tag{1.7.17}$$

Using expressions (1.7.15), we find that

$$\mathrm{d}\mathbf{e}_r = \mathrm{d}\theta\,\mathbf{e}_\theta + \sin\theta\,\mathrm{d}\varphi\,\mathbf{e}_\varphi. \tag{1.7.18}$$

Substituting into (1.7.17), we obtain

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\,\sin\theta\,\mathrm{d}\varphi\,\mathbf{e}_\varphi. \tag{1.7.19}$$

The fundamental form of space takes the form

$$\mathrm{d}^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = (\mathrm{d}r)^2 + r^2\,(\mathrm{d}\theta)^2 + r^2\,\sin^2\theta\,(\mathrm{d}\varphi)^2. \tag{1.7.20}$$

The associated metric coefficients are 1, r^2, $r^2\sin^2\theta$.

An arbitrary vector, $\mathbf{v}$, can be resolved as

$$\mathbf{v} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_\varphi\,\mathbf{e}_\varphi, \tag{1.7.21}$$

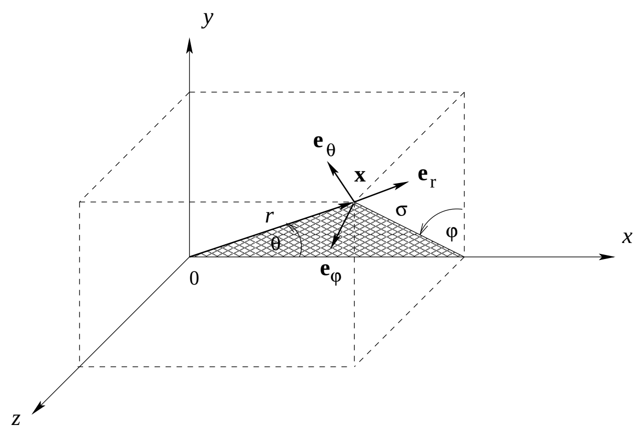

FIGURE 1.7.2 Illustration of spherical polar coordinates, (r, θ, φ), defined with respect to the Cartesian coordinates, (x, y, z), and cylindrical polar coordinates, (x, σ, φ), where r is the distance from the origin, θ is the meridional angle, φ is the azimuthal angle, and σ is the distance from the x axis.

where v_r, v_θ, and v_φ are the spherical polar components of the underlying vector, $\mathbf{v}$, and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ are position-dependent mutually orthogonal unit vectors.

The tensor product of two vectors, $\mathbf{v}$ and $\mathbf{u}$, is

$$\mathbf{T} = v_\alpha u_\beta \, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta, \tag{1.7.22}$$

where summation of the Greek indices is implied in the range r, θ, φ. For example,

$$\mathbf{e}_r \otimes \mathbf{e}_r = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta\cos\varphi & \cos\theta\sin\theta\sin\varphi \\ \cos\theta\sin\theta\cos\varphi & \sin^2\theta\cos^2\varphi & \sin^2\theta\sin\varphi\cos\varphi \\ \cos\theta\sin\theta\sin\varphi & \sin^2\theta\sin\varphi\cos\varphi & \sin^2\theta\sin^2\varphi \end{bmatrix} \tag{1.7.23}$$

is a dense symmetric matrix. Eight similar tensor products of unit vectors generate similar matrices.

Exercise

1.7.1 Formulate the base matrix $\mathbf{e}_\sigma \otimes \mathbf{e}_\sigma$ in cylindrical polar coordinates.

1.8 Change of Cartesian base

Let $\mathbf{e}_i$ be a set of mutually orthogonal unit vectors defining an arbitrary Cartesian base, so that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where δ_{ij} is Kronecker's delta. Also let $\widetilde{\mathbf{e}}_i$ be another set of mutually orthogonal unit vectors defining another a Cartesian base, so that $\widetilde{\mathbf{e}}_i \cdot \widetilde{\mathbf{e}}_j = \delta_{ij}$. One of the two bases could be the universal base, $\boldsymbol{\epsilon}_i$.

1.8.1 Transformation matrix

The second set of base vectors, denoted by a tilde, can be related to the first set by a linear expansion,

$$\widetilde{\mathbf{e}}_i = Q_{ij}\, \mathbf{e}_j, \tag{1.8.1}$$

where $\mathbf{Q}$ is a transformation matrix. The matrix $\mathbf{Q}$ is the counterpart of the less specific matrix $\mathbf{H}$ introduced in (1.3.3). The change of notation is motivated by the orthogonality of $\mathbf{Q}$, demonstrated next.

Taking the inner product of both sides of equation (1.8.1) with $\mathbf{e}_m$ or $\widetilde{\mathbf{e}}_m$, where m is a free index, we find that

$$Q_{im} \equiv \widetilde{\mathbf{e}}_i \cdot \mathbf{e}_m, \qquad \delta_{im} = Q_{ij} Q_{mj} = Q_{ij} Q^{\mathrm{T}}_{jm}. \tag{1.8.2}$$

The second equation shows that the transformation matrix is orthogonal,

$$\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}, \tag{1.8.3}$$

where the superscript -1 denotes the matrix inverse, the superscript T denotes the matrix transpose, and the superscript $-\mathrm{T}$ denotes the inverse of the transpose. Consequently, equation (1.8.1) can be inverted readily to yield

$$\mathbf{e}_i = Q_{ji}\, \widetilde{\mathbf{e}}_j. \tag{1.8.4}$$

Note the order of the indices on the right-hand side.

1.8.2 Numerical confirmation of transformation rules

The following Matlab code named *cartesian*, located in directory VECAR of TUNLIB, confirms these transformation rules:

```
%---
% first base
%---

thet1 = 0.0845*pi;               % arbitrary
thet2 = thet1 + 0.5*pi

e1(1) = cos(thet1); e1(2) = sin(thet1);
e2(1) = cos(thet2); e2(2) = sin(thet2);

%---
% second base (tilded)
%---

thett1 = 0.1234*pi;            % arbitrary
thett2 = thett1 + 0.5*pi;

et1(1) = cos(thett1); et1(2) = sin(thett1);
et2(1) = cos(thett2); et2(2) = sin(thett2);

%---
% transformation matrix
%---

Q(1,1) = et1(1)*e1(1) + et1(2)*e1(2);
Q(1,2) = et1(1)*e2(1) + et1(2)*e2(2);
Q(2,1) = et2(1)*e1(1) + et2(2)*e1(2);
Q(2,2) = et2(1)*e2(1) + et2(2)*e2(2);

%---
% conversions
%---
```

```
E = [ e1(1) e2(1);
      e1(2) e2(2)];

Et = [ et1(1) et2(1);
       et1(2) et2(2)];

[Et E*Q']
[E  Et*Q]
```

The prime in the penultimate line denotes the matrix transpose. Running the code generates the following output, as instructed by the last two lines of the code:

```
    0.9258   -0.3780    0.9258   -0.3780
    0.3780    0.9258    0.3780    0.9258

    0.9650   -0.2624    0.9650   -0.2624
    0.2624    0.9650    0.2624    0.9650
```

We note that, as expected, the first and second pairs of columns are identical.

1.8.3 Determinant

Taking the determinant of (1.8.3), we find that

$$\det^2(\mathbf{Q}) = 1. \tag{1.8.5}$$

In fact, since the vector base conforms with the right-handed rule, $\det(\mathbf{Q}) = 1$.

In two dimensions, if the tilde system arises by rotating the untilded system around the origin by angle ϱ, then

$$\mathbf{Q} \equiv \begin{bmatrix} \cos\varrho & \sin\varrho \\ -\sin\varrho & \cos\varrho \end{bmatrix}. \tag{1.8.6}$$

The determinant of this matrix is readily confirmed to be unity. When $\varrho = \pi$, we find that $\mathbf{Q} = -\mathbf{I}$.

1.8.4 Base transformation rules

The vectors $\mathbf{e}_i$ can be arranged at the *columns* of a matrix, $\mathbf{E}$, and the vectors $\widetilde{\mathbf{e}}_i$ can be arranged at the *columns* of another matrix, $\widetilde{\mathbf{E}}$,

$$\mathbf{E} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \cdots & \mathbf{e}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \qquad \widetilde{\mathbf{E}} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \widetilde{\mathbf{e}}_1 & \cdots & \widetilde{\mathbf{e}}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \tag{1.8.7}$$

where $\mathbf{E}^{\mathrm{T}} \cdot \mathbf{E} = \mathbf{I}$ and $\widetilde{\mathbf{E}}^{\mathrm{T}} \cdot \widetilde{\mathbf{E}} = \mathbf{I}$. Using (1.8.1) and (1.8.4), we find that

$$\widetilde{\mathbf{E}} = \mathbf{E} \cdot \mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{E} = \widetilde{\mathbf{E}} \cdot \mathbf{Q}, \qquad \mathbf{Q} = \widetilde{\mathbf{E}}^{\mathrm{T}} \cdot \mathbf{E}. \tag{1.8.8}$$

If the matrix $\mathbf{Q}$ were not orthogonal, the inverse instead of the transpose would appear on the right-hand side of the first relation.

1.8.5 Transformation of vector components

An arbitrary vector, $\mathbf{v}$, can be expanded in each set of base vectors as

$$\mathbf{v} = c_j \, \mathbf{e}_j = \widetilde{c}_j \, \widetilde{\mathbf{e}}_j, \tag{1.8.9}$$

where summation is implied over the repeated index, j.

Projecting equation (1.8.9) onto $\widetilde{\mathbf{e}}_i$, where i is a free index, we obtain the tilded components,

$$\widetilde{c}_i = c_j \, (\widetilde{\mathbf{e}}_i \cdot \mathbf{e}_j), \tag{1.8.10}$$

where summation is implied over the repeated index, j. Projecting equation (1.8.9) onto $\mathbf{e}_i$, we find that the untilded vector components are given by

$$c_i = \widetilde{c}_j \, (\mathbf{e}_i \cdot \widetilde{\mathbf{e}}_j), \tag{1.8.11}$$

where summation is implied over the repeated index, j. We have found that

$$\widetilde{c}_i = Q_{ij} \, c_j, \qquad c_i = Q_{ji} \, \widetilde{c}_j = \widetilde{c}_j Q_{ji} = Q^{\mathrm{T}}_{ij} \, \widetilde{c}_j, \tag{1.8.12}$$

where the superscript T denotes the matrix transpose. In vector notation,

$$\widetilde{\mathbf{c}} = \mathbf{Q} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}. \tag{1.8.13}$$

These relations confirm further that the matrix $\mathbf{Q}$ is orthogonal, that is, its inverse is equal to its transpose.

Invoking (1.4.15), we write

$$\mathbf{c} = \mathbf{E}^{\mathrm{T}} \cdot \mathbf{v}, \qquad \widetilde{\mathbf{c}} = \widetilde{\mathbf{E}}^{\mathrm{T}} \cdot \mathbf{v}. \tag{1.8.14}$$

Combining these equations we find that

$$\widetilde{\mathbf{c}} = \widetilde{\mathbf{E}}^{\mathrm{T}} \cdot \mathbf{E} \cdot \mathbf{c}, \tag{1.8.15}$$

which is consistent with (1.8.13).

1.8.6 First-order tensors

If the components of a vector conform with the transformation rules (1.8.12), then the vector is accepted as a *first-order* tensor. If they do not, the vector is regarded as a mere numerical array.

In fact, the transformation rules (1.8.12) are specializations of those shown in (1.3.12), summarized below for convenience,

$$\widetilde{\mathbf{c}} = \mathbf{H}^{-\mathrm{T}} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{H}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}. \tag{1.8.16}$$

The transformation rules (1.8.12) arise from those shown in (1.8.16) by setting $\mathbf{H} = \mathbf{Q}$ and recalling that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$, and thus $\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$.

The differential displacement, $\mathrm{d}\mathbf{x}$, is a vector that qualifies as a first-order tensor, where $\mathbf{x}$ is position in space. The proof relies on elementary trigonometry.

Consequently, the velocity of a particle, $\mathbf{v} = \mathrm{d}\mathbf{X}/\mathrm{d}t$, is also a first-order tensor, where $\mathbf{X}$ is the particle position and t stands for time. In contrast, the vectorial array $\mathbf{w} = [v_x^p, v_y^q, v_z^s]$ is a first-order tensor only when $p = q = s = 1$.

1.8.7 Transformation matrix in terms of coordinates

Now we consider the Cartesian coordinates of the position in the untilded and tilded bases, X_i and $\widetilde{X}_i$, and express the position as

$$\mathbf{x} = \mathbf{x}_0 + X_i\,\mathbf{e}_i, \qquad \mathbf{x} = \widetilde{\mathbf{x}}_0 + \widetilde{X}_i\,\widetilde{\mathbf{e}}_i, \tag{1.8.17}$$

where $\mathbf{x}_0$ is the position of the origin. Differentiating these expressions, or else using the chain rule, we obtain

$$\widetilde{\mathbf{e}}_i \equiv \frac{\partial \mathbf{x}}{\partial \widetilde{X}_i} = \frac{\partial X_j}{\partial \widetilde{X}_i}\frac{\partial \mathbf{x}}{\partial X_j} = \frac{\partial X_j}{\partial \widetilde{X}_i}\,\mathbf{e}_j \tag{1.8.18}$$

and

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{x}}{\partial X_i} = \frac{\partial \widetilde{X}_j}{\partial X_i}\frac{\partial \mathbf{x}}{\partial \widetilde{X}_j} = \frac{\partial \widetilde{X}_j}{\partial X_i}\,\widetilde{\mathbf{e}}_j. \tag{1.8.19}$$

Projecting equation (1.8.18) onto $\mathbf{e}_m$ or $\widetilde{\mathbf{e}}_m$, we obtain

$$Q_{im} = \frac{\partial X_m}{\partial \widetilde{X}_i}, \qquad \delta_{im} = \frac{\partial X_j}{\partial \widetilde{X}_i}\,Q_{mj}, \tag{1.8.20}$$

where m is a free index. Also projecting equation (1.8.18) onto $\widetilde{\mathbf{e}}_m$ or $\mathbf{e}_m$, we obtain

$$Q_{mi} = \frac{\partial \widetilde{X}_m}{\partial X_i}, \qquad \delta_{im} = \frac{\partial \widetilde{X}_j}{\partial X_i}\,Q_{jm}. \tag{1.8.21}$$

Renaming the indices in the last two sets of equations, we find that

$$Q_{ij} = \frac{\partial X_j}{\partial \widetilde{X}_i} = \frac{\partial \widetilde{X}_i}{\partial X_j} \tag{1.8.22}$$

and

$$Q^{-1}_{ij} = \frac{\partial X_i}{\partial \widetilde{X}_j} = \frac{\partial \widetilde{X}_j}{\partial X_i} = Q_{ji} = Q^{\mathrm{T}}_{ij}, \tag{1.8.23}$$

thereby confirming the orthogonality of the transformation matrix, $\mathbf{Q}$. From these relations, we find that

$$\begin{aligned} Q_{ij}Q^{-1}_{jk} &= \frac{\partial X_j}{\partial \widetilde{X}_i}\,\frac{\partial \widetilde{X}_k}{\partial X_j} = \frac{\partial \widetilde{X}_i}{\partial X_j}\,\frac{\partial X_j}{\partial \widetilde{X}_k} \\ &= \frac{\partial X_j}{\partial \widetilde{X}_i}\,\frac{\partial X_j}{\partial \widetilde{X}_k} = \frac{\partial \widetilde{X}_i}{\partial X_j}\,\frac{\partial \widetilde{X}_k}{\partial X_j} = \delta_{ik}. \end{aligned} \tag{1.8.24}$$

1.8.8 Transformation rules in terms of coordinates

In terms of coordinate derivatives, the transformation rules derived previously in this section for the base vectors take the form

$$\widetilde{\mathbf{b}}_i = \frac{\partial X_j}{\partial \widetilde{X}_i}\,\mathbf{b}_j = \frac{\partial \widetilde{X}_i}{\partial X_j}\,\mathbf{b}_j \tag{1.8.25}$$

and

$$\mathbf{b}_i = \frac{\partial X_i}{\partial \widetilde{X}_j}\,\widetilde{\mathbf{b}}_j = \frac{\partial \widetilde{X}_j}{\partial X_i}\,\widetilde{\mathbf{b}}_j. \tag{1.8.26}$$

The associated transformation rules for the vector components take the form

$$\widetilde{c}_i = \frac{\partial X_j}{\partial \widetilde{X}_i}\,c_j = \frac{\partial \widetilde{X}_i}{\partial \widetilde{X}_j}\,c_j \tag{1.8.27}$$

and

$$c_i = \frac{\partial X_i}{\partial \widetilde{X}_j}\,\widetilde{c}_j = \frac{\partial \widetilde{X}_j}{\partial X_i}\,\widetilde{c}_j. \tag{1.8.28}$$

When referring to the universal base, X can be written as x.

Exercise

1.8.1 Show that the outer product of two vectors is a first-order tensor.

1.9 Zeroth-order tensors

Assume that a particle is located at position $\mathbf{x}$ and another particle is located at position $\mathbf{y}$ in N-dimensional space. The particle coordinates in the universal Cartesian base are encapsulated in the N-dimensional vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}. \tag{1.9.1}$$

The square of the distance between the two particles is

$$|\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = (x_i - y_i)(x_i - y_i), \tag{1.9.2}$$

where summation is implied over the repeated index, i.

1.9.1 First base

The coordinates of the first particle in an arbitrary Cartesian base $\mathbf{e}_i$ are denoted by c_i, and the coordinates of the second particle in the same base are denoted by d_i, so that

$$\mathbf{x} = c_i\,\mathbf{e}_i, \qquad \mathbf{y} = d_i\,\mathbf{e}_i. \tag{1.9.3}$$

We find that

$$|\mathbf{x}-\mathbf{y}|^2 = (c_i - d_i)\,\mathbf{e}_i\cdot(c_j - d_j)\,\mathbf{e}_j = (c_i - d_i)(c_i - d_i), \tag{1.9.4}$$

where summation is implied over the repeated index, i.

1.9.2 Second base

The coordinates of the first particle in another arbitrary Cartesian base $\widetilde{\mathbf{e}}_i$ are $\widetilde{c}_i$, and the coordinates of the second particle in the same base are $\widetilde{d}_i$, so that

$$\mathbf{x} = \widetilde{c}_i\,\widetilde{\mathbf{e}}_i, \qquad \mathbf{y} = \widetilde{d}_i\,\widetilde{\mathbf{e}}_i. \tag{1.9.5}$$

We find that

$$|\mathbf{x}-\mathbf{y}|^2 = (\widetilde{c}_i - \widetilde{d}_i)\,\mathbf{e}_i\cdot(\widetilde{c}_j - \widetilde{d}_j)\,\mathbf{e}_j = (\widetilde{c}_i - \widetilde{d}_i)(\widetilde{c}_i - \widetilde{d}_i), \tag{1.9.6}$$

where summation is implied over the repeated index, i.

1.9.3 Invariance

Now using the component transformation rules

$$\widetilde{c}_i = Q_{ij}\,c_j, \qquad \widetilde{d}_i = Q_{ij}\,d_j, \tag{1.9.7}$$

we find that

$$(\widetilde{c}_i - \widetilde{d}_i)(\widetilde{c}_i - \widetilde{d}_i) = Q_{ij}\,(c_j - d_j)\,Q_{im}\,(c_m - d_m). \tag{1.9.8}$$

Rearranging, we obtain

$$(\widetilde{c}_i - \widetilde{d}_i)(\widetilde{c}_i - \widetilde{d}_i) = (c_j - d_j)\, Q^{\mathrm{T}}_{ji}\, Q_{im}\,(c_m - d_m). \tag{1.9.9}$$

Since the matrix $\mathbf{Q}$ is orthogonal, $Q^{\mathrm{T}}_{ji}\, Q_{im} = \delta_{jm}$, and thus

$$(\widetilde{c}_i - \widetilde{d}_i)(\widetilde{c}_i - \widetilde{d}_i) = (c_i - d_i)(c_i - d_i), \tag{1.9.10}$$

which shows that the distance between the two particles predicted from (1.9.6) is the same as that predicted from (1.9.4). We say that the distance is a zeroth-order tensor.

A zeroth-order tensor is a scalar whose value is independent of the chosen frame of reference. Examples are the inner product of two vectors, the temperature of a point in a medium, but not necessarily the color of a star.

Exercise

1.9.1 Show that the inner product of two vectors is a zeroth-order tensor.

1.10 Matrix bases and matrix components

An $M \times N$ matrix is a collection of numbers arranged in a table with M rows and N columns. An example is the 2×3 matrix

$$\mathbf{T} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 9 & 11 \end{bmatrix}. \tag{1.10.1}$$

In the case of a square matrix, $M = N$. In the case of a vertical array matrix, $N = 1$. In the case of a horizontal array matrix, $M = 1$.

1.10.1 Matrix elements

The *elements* of a matrix, $\mathbf{T}$, are denoted by T_{ij}, where $i = 1, \ldots, M$ and $j = 1, \ldots, N$. With reference to the matrix shown in (1.10.1), $T_{12} = 3$ and $T_{21} = 7$. The total number of elements of an $M \times N$ matrix is MN. The total number of elements of a square matrix is N^2.

We will refer to the *elements* of a matrix also as *entries*, but *not* components. The term *components* will be reserved for the matrix coefficients in a specified matrix base.

1.10.2 A matrix contains arrays

An $M \times N$ matrix can be regarded either as an ordered collection of M horizontal N-dimensional arrays placed in its rows, or as an ordered collection of N vertical M-dimensional arrays in its columns. This interpretation leads us to the notion of second-order tensors defined as matrices that obey appropriate coordinate or base transformation rules based on those for the constituent vectors.

1.10.3 Expansion of a square matrix

Consider an arbitrary square $N \times N$ matrix, $\mathbf{T}$, and introduce a set of N^2 arbitrary $N \times N$ matrices, $\mathbf{B}_{ij}$ for $i, j = 1, \ldots, N$, comprising a *matrix base*. Unless the base is chosen poorly, we will able to expand the matrix $\mathbf{T}$ into a weighted sum using the usual rules of matrix algebra,

$$\mathbf{T} = \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}\, \mathbf{B}_{ij}, \tag{1.10.2}$$

where C_{ij} is a collection of N^2 coefficients. Explicitly,

$$\mathbf{T} = C_{11}\, \mathbf{B}_{11} + C_{12}\, \mathbf{B}_{12} + \cdots + C_{NN}\, \mathbf{B}_{NN}\, . \tag{1.10.3}$$

We recall that, to multiply a matrix by a scalar, we multiply each scalar element. To add two or any number of matrices, we add corresponding matrix elements.

1.10.4 Einstein summation convention

We recall the Einstein summation convention stipulating that *if an index appears twice in a product, summation over that index is implied.* The same index may not appear more than twice. An index that appears once is a free index. Under the Einstein summation convention, expansion (1.10.3) takes the simple form

$$\mathbf{T} = C_{ij}\mathbf{B}_{ij}, \tag{1.10.4}$$

where double summation is implied over the repeated indices, i and j, in their entire range, $i, j = 1, \ldots, N$.

1.10.5 Matrix transpose

The transpose of the matrix $\mathbf{T}$, indicated by the superscript T, is given by

$$\mathbf{T}^{\mathrm{T}} = C_{ij} \mathbf{B}_{ij}^{\mathrm{T}}. \tag{1.10.5}$$

Note that $C_{ij} = C_{ji}$ does not guarantee that $\mathbf{T}$ is symmetric, that is, $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$.

1.10.6 Matrix elements v. matrix components

The N^2 scalar coefficients, C_{ij}, are the *components* of the matrix $\mathbf{T}$ in the specified matrix base. By contrast, the numbers T_{ij} are the elements of the matrix $\mathbf{T}$. The former can be arranged in a square $N \times N$ component matrix, $\mathbf{C}$, which is generally different than the matrix $\mathbf{T}$. Only if all elements of each base matrix $\mathbf{B}_{ij}$ are zero, except for the ij element that is equal to unity, is the component matrix $\mathbf{C}$ identical to the given matrix $\mathbf{T}$.

The component matrix $\mathbf{C}$ of a given matrix $\mathbf{T}$ depends on the chosen matrix base, $\mathbf{B}_{ij}$. We will see that a real or complex matrix base can be found where the component matrix $\mathbf{C}$ is diagonal.

1.10.7 Computation of matrix components

To compute the N^2 matrix components, C_{ij}, corresponding to a specified base, we enforce the matrix equation (1.10.4) for each matrix element and derive a system of N^2 linear equations.

As an example, we consider the 2×2 matrix

$$\mathbf{T} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \tag{1.10.6}$$

and introduce a matrix base consisting of four matrices,

$$\mathbf{B}_{11} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{B}_{12} = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$
$$\mathbf{B}_{21} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad \mathbf{B}_{22} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \qquad (1.10.7)$$

To compute the four components, C_{ij}, we enforce the representation (1.10.4) for each element of $\mathbf{T}$ and derive a system of four linear equations. The solution is computed by the following Matlab code named *base1*, located in directory TENBASE of TUNLIB:

```
RHS = [3 4 1 2];

MAT = 0.5*[ 2 1 1 1;
            1 2 3 1;
            1 1 2 1;
            1 1 1 4];

SOL = RHS/MAT'
```

The prime in the last line of the code denotes the matrix transpose. Running the code generates the following output:

```
    0.2857    9.7143   -3.7143   -0.5714
```

Based on these results, we set

$$C_{11} = 0.2857, \quad C_{12} = 9.7143,$$
$$C_{21} = -3.7143, \quad C_{22} = -0.5714, \qquad (1.10.8)$$

and compile the component matrix corresponding to the specified base,

$$\mathbf{C} = \begin{bmatrix} 0.2857 & 9.7143 \\ -3.7143 & -0.5714 \end{bmatrix}. \qquad (1.10.9)$$

This matrix differs from the matrix $\mathbf{T}$ shown in (1.10.6).

1.10.8 Object described by components

We may regard a given matrix $\mathbf{T}$ as an object, and the component matrix $\mathbf{C}$ as the encapsulated attributes (fingerprint) of the object in a specified matrix base representing a viewpoint. An object could be a car in a dealer's parking lot, and the components could the car's parts in the dealer's service manual.

1.10.9 Expansions in different bases

Since a given matrix, $\mathbf{T}$, has different component matrices, $\mathbf{C}$, in different bases, we may write

$$\mathbf{T} = C_{ij}\,\mathbf{B}_{ij} = \widetilde{C}_{ij}\,\widetilde{\mathbf{B}}_{ij}, \tag{1.10.10}$$

where C_{ij} are the components of $\mathbf{T}$ in the $\mathbf{B}_{ij}$ base, $\widetilde{C}_{ij}$ are the components of $\mathbf{T}$ in the $\widetilde{\mathbf{B}}_{ij}$ base, and double summation is implied over the repeated indices, i and j.

The base matrices can be related by a linear transformation involving appropriate coefficients, $A_{ijk\ell}$,

$$\mathbf{B}_{ij} = A_{ijk\ell}\,\widetilde{\mathbf{B}}_{k\ell}, \tag{1.10.11}$$

where double summation is implied over the repeated indices, k and ℓ. Substituting this expression into (1.10.10), we obtain

$$C_{ij}A_{ijk\ell}\,\mathbf{B}_{k\ell} = \widetilde{C}_{k\ell}\,\mathbf{B}_{k\ell}, \tag{1.10.12}$$

which shows that

$$\widetilde{C}_{k\ell} = C_{ij}\,A_{ijk\ell}, \tag{1.10.13}$$

where k and ℓ are free indices and double summation is implied over the repeated indices, i and j.

1.10.10 Second-order tensors in terms of base components

If the components of a matrix conform with the transformation rule (1.10.13), then the matrix is accepted as a *second-order* tensor. Examples will be given later in this chapter with reference to a pair of Cartesian bases.

Exercises

1.10.1 Propose a base suitable for a 2×3 matrix.

1.10.2 Compute the components of the matrix $\mathbf{T}$ given in (1.10.6) for a base consisting of the matrices

$$[\mathbf{B}_{ij}]_{k\ell} = \sqrt{k+i}/\ln(\ell + j), \tag{1.10.14}$$

where $[\mathbf{B}_{ij}]_{k\ell}$ is the $k\ell$ element of $\mathbf{B}_{ij}$.

1.11 Dyadic matrix base

We may select N linearly independent N-dimensional arrays,

$$\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N, \tag{1.11.1}$$

and formulate a matrix base as a *dyadic product*,

$$\mathbf{B}_{ij} = \mathbf{b}_i \otimes \mathbf{b}_j, \tag{1.11.2}$$

where $\otimes$ denotes the *dyadic product*. This notation means that

$$[\mathbf{B}_{ij}]_{k\ell} = [\mathbf{b}_i]_k \times [\mathbf{b}_j]_\ell, \tag{1.11.3}$$

where $[\mathbf{B}_{ij}]_{k\ell}$ is the $k\ell$th element of $\mathbf{B}_{ij}$, $[\mathbf{b}_i]_k$ is the kth element of $\mathbf{b}_i$, $[\mathbf{b}_j]_\ell$ is the ℓth element of $\mathbf{b}_j$, and $\times$ denotes regular scalar multiplication. The dyadic product is also known as the *tensor product* or *Cartesian product*, as discussed in Section 1.6.

As an example, we choose the vectors $\mathbf{b}_1 = [1, 0]$ and $\mathbf{b}_2 = [1, 1]$, and formulate the dyadic matrix base

$$\mathbf{B}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{B}_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B}_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1.11.4}$$

Because $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent, this is a perfectly acceptable base.

1.11.1 Dyadic expansion

The general decomposition (1.10.3) of an $N \times N$ matrix, $\mathbf{T}$, subject to the dyadic base defined in (1.11.2) takes the form

$$\mathbf{T} = C_{ij}\,\mathbf{b}_i \otimes \mathbf{b}_j, \tag{1.11.5}$$

where summation is implied over the repeated indices, i and j.

1.11.2 Matrix transpose

The matrix transpose is given by the expansion

$$\mathbf{T}^{\mathrm{T}} = C_{ji}\,\mathbf{b}_i \otimes \mathbf{b}_j. \tag{1.11.6}$$

When $C_{ij} = C_{ji}$, the matrix $\mathbf{T}$ is symmetric. When $C_{ij} = -C_{ji}$, the matrix $\mathbf{T}$ is antisymmetric.

1.11.3 Computation of matrix components

To compute the matrix components, C_{ij}, we may enforce the representation (1.11.5) for each element of $\mathbf{T}$ and derive a system of linear equations. As an example, we consider the 2×2 matrix

$$\mathbf{T} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \tag{1.11.7}$$

and adopt the base stated in (1.11.4). The solution of the linear system is computed by the following Matlab code named *base2*, located in directory Tenbase of Tunlib:

```
RHS = [3 4 1 2];

MAT = [ 1 1 1 1;
        0 1 0 1;
        0 0 1 1;
        0 0 0 1];

SOL = RHS/MAT'
```

The prime in the last line of the code denotes the matrix transpose. Running the code generates the following output:

```
     0     2    -1     2
```

Based on these results, we set

$$C_{11} = 0, \qquad C_{12} = 2, \qquad C_{21} = -1, \qquad C_{22} = 2, \quad (1.11.8)$$

and formulate the component matrix

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}, \qquad (1.11.9)$$

which differs from the matrix $\mathbf{T}$ shown in (1.10.6). If we had chosen $\mathbf{b}_1 = [1, 0]$ and $\mathbf{b}_2 = [0, 1]$, we would have found that $\mathbf{C} = \mathbf{T}$.

A more general inclusive Matlab code named *base3*, located in directory TENBASE of TUNLIB, reads:

```
RHS = [3 4 1 2];

b1(1) = 1.0; b1(2) = 0.0;
b2(1) = 1.0; b2(2) = 1.0;

MAT = ...
  ...
[ b1(1)*b1(1) , b1(1)*b2(1) , b2(1)*b1(1) , b2(1)*b2(1);
  b1(1)*b1(2) , b1(1)*b2(2) , b2(1)*b1(2) , b2(1)*b2(2);
  b1(2)*b1(1) , b1(2)*b2(1) , b2(2)*b1(1) , b2(2)*b2(1);
  b1(2)*b1(2) , b1(2)*b2(2) , b2(2)*b1(2) , b2(2)*b2(2)];

SOL = RHS/MAT'
```

The dyadic products are implemented directly into this code.

1.11.4 Components by projection

In an alternative formulation, we project the expansion (1.11.5) from the left onto $\mathbf{b}_m$, where m is a free index, and obtain

$$\mathbf{b}_m \cdot \mathbf{T} = b_{mi} \, C_{ij} \, \mathbf{b}_j, \qquad (1.11.10)$$

where summation is implied over the repeated indices i and j, and

$$b_{mi} \equiv \mathbf{b}_m \cdot \mathbf{b}_i \tag{1.11.11}$$

are base vector projections called metric coefficients. In index notation, equation (1.11.11) takes the form

$$b_{mi} = [\mathbf{b}_m]_k [\mathbf{b}_i]_k, \tag{1.11.12}$$

where summation is implied over the repeated index, k. In index notation, equation (1.11.10) takes the form

$$[\mathbf{b}_m]_k \, [\mathbf{T}]_{k\ell} = b_{mi} \, C_{ij} \, [\mathbf{b}_j]_\ell, \tag{1.11.13}$$

where ℓ is a free index and summation is implied over the repeated indices i, j, and k.

Projecting expansion (1.11.5) from the right onto $\mathbf{b}_m$, we obtain the companion equation

$$\mathbf{T} \cdot \mathbf{b}_m = b_{jm} \, C_{ij} \, \mathbf{b}_i, \tag{1.11.14}$$

where m is a free index.

Now projecting (1.11.10) onto $\mathbf{b}_n$, we obtain

$$\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n = b_{mi} \, b_{jn} \, C_{ij}, \tag{1.11.15}$$

where n is a free index. Projecting also (1.11.14) onto $\mathbf{b}_n$, where n is a free index, we obtain

$$\mathbf{b}_n \cdot \mathbf{T} \cdot \mathbf{b}_m = b_{jm} \, b_{ni} \, C_{ij}, \tag{1.11.16}$$

which is nothing but (1.11.15) with the indices m and n transposed.

Applying (1.11.15) for $m, n = 1, \ldots, N$ provides us with a system of linear equations for the elements of the component matrix, T_{ij}. For example, for $N = 2$, $m = 1$, and $n = 1$, we obtain

$$\mathbf{b}_1 \cdot \mathbf{T} \cdot \mathbf{b}_1 = b_{11} \, b_{11} \, C_{11} + b_{11} \, b_{12} \, C_{12} + b_{12} \, b_{11} \, C_{21} + b_{12} \, b_{12} \, C_{22}, \tag{1.11.17}$$

involving all elements of the component matrix, $\mathbf{C}$.

1.11.5 base4

The following Matlab code named *base4*, located in directory TENBASE of TUNLIB, compiles and solves the linear system:

```
T = [3 4; 1 2];

b1(1) = 1.0; b1(2) = 0.0;   % base arrays (arbitrary)
b2(1) = 1.0; b2(2) = 1.0;   % base arrays (arbitrary)

%---
% metric tensor
%---

bmet(1,1) =  b1*b1'; bmet(1,2) =  b1*b2';
bmet(2,1) =  b2*b1'; bmet(2,2) =  b2*b2';

G = bmet;

%---
% system is MAT * SOL = RHS
%---

RHS(1) = b1*T*b1'; RHS(2) = b1*T*b2';
RHS(3) = b2*T*b1'; RHS(4) = b2*T*b2';

MAT(1,1) = G(1,1)*G(1,1); MAT(1,2) = G(1,1)*G(2,1);
MAT(1,3) = G(1,2)*G(1,1); MAT(1,4) = G(1,2)*G(2,1);

MAT(2,1) = G(1,1)*G(1,2); MAT(2,2) = G(1,1)*G(2,2);
MAT(2,3) = G(1,2)*G(1,2); MAT(2,4) = G(1,2)*G(2,2);

MAT(3,1) = G(2,1)*G(1,1); MAT(3,2) = G(2,1)*G(2,1);
MAT(3,3) = G(2,2)*G(1,1); MAT(3,4) = G(2,2)*G(2,1);

MAT(4,1) = G(2,1)*G(1,2); MAT(4,2) = G(2,1)*G(2,2);
MAT(4,3) = G(2,2)*G(1,2); MAT(4,4) = G(2,2)*G(2,2);

SOL = RHS/MAT'
```

Running the code generates the following expected output:

```
     0         2        -1         2
```

which is consistent with data obtained previously using a different method. The advantages of this formulation will be discussed in Section 1.3 with reference to a Cartesian matrix base.

Exercise

1.11.1 Compute the components of the matrix $\mathbf{T}$ given in (1.10.6) for a dyadic matrix base with $\mathbf{b}_1 = [1, 0.1]$ and $\mathbf{b}_2 = [0.1, 1]$.

1.12 Cartesian tensor base

Assume that the dyadic base arrays are orthonormal, denoted by $\mathbf{b}_i = \mathbf{e}_i$ for $i = 1, \ldots, N$, where $|\mathbf{e}_i| = 1$ and $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$ so that

$$\mathbf{e}_p \cdot \mathbf{e}_q = \delta_{pq}, \tag{1.12.1}$$

where δ_{pq} is Kronecker's delta: $\delta_{pq} = 1$ if $p = q$ or 0 otherwise. Consequently, $\mathbf{b}$ is the identity matrix, $\mathbf{I}$.

1.12.1 Cartesian dyadic matrix base

The corresponding dyadic matrix base, $\mathbf{B}_{ij}$, is a *Cartesian matrix base* denoted by

$$\mathbf{E}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.12.2}$$

By definition, the $k\ell$ element of $\mathbf{E}_{ij}$ is given by

$$[\mathbf{E}_{ij}]_{k\ell} = [\mathbf{e}_i]_k \times [\mathbf{e}_j]_\ell, \tag{1.12.3}$$

where $\times$ denotes regular scalar multiplication. Note that

$$\mathbf{E}_{ij} = \mathbf{E}_{ji}^{\mathrm{T}}, \tag{1.12.4}$$

where the superscript T denotes the transpose.

For example, in three dimensions, $N = 3$, we may choose

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{1.12.5}$$

to find that

$$\mathbf{E}_{11} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{E}_{12} = \tfrac{1}{2} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.12.6}$$

and seven other dyadic base matrices. Note that the dyadic base matrices are not necessarily sparse.

1.12.2 Cartesian matrix components

Because of (1.12.1), equation (1.11.15) provides us with an explicit expression for the matrix components,

$$C_{mn} = \mathbf{e}_m \cdot \mathbf{T} \cdot \mathbf{e}_n = \mathbf{T} : (\mathbf{e}_m \otimes \mathbf{e}_n). \tag{1.12.7}$$

The computation of the first expression requires a matrix-vector followed by a vector–vector multiplication.

1.12.3 Identity matrix

The identity matrix, $\mathbf{I}$, is distinguished by the property that $\mathbf{I} \cdot \mathbf{a} = \mathbf{a}$, for any vector, $\mathbf{a}$. We may confirm readily that the components of the identity matrix in any Cartesian base are $C_{mn} = \delta_{mn}$, Consequently,

$$\mathbf{I} = \mathbf{E}_{ii} \equiv \mathbf{E}_{11} + \cdots + \mathbf{E}_{NN} = \mathbf{e}_i \otimes \mathbf{e}_i, \tag{1.12.8}$$

where summation is implied over the repeated index, i.

1.12.4 Confirmation by code

The following Matlab code named *cartesian*, located in directory TENCAR of TUNLIB, displays a Cartesian dyadic base and confirms identity (1.12.8) in two dimensions:

```
%---
% construct a 2x2 Cartesian base
%---

theta1 = 0.2345*pi;         % arbitrary
theta2 = theta1+0.5*pi;     % orthogonal

e1(1) = cos(theta1); e1(2) = sin(theta1);
e2(1) = cos(theta2); e2(2) = sin(theta2);

for i=1:2
 for j=1:2
  E11(i,j) = e1(i)*e1(j);
  E12(i,j) = e1(i)*e2(j);
  E21(i,j) = e2(i)*e1(j);
  E22(i,j) = e2(i)*e2(j);
 end
end

%---
% display the base
%---

[E11 E12]
[E21 E22]

%---
% Identity matrix
%---

E11+E22
```

Running the code generates the following output:

```
    0.5486    0.4976   -0.4976    0.5486
    0.4976    0.4514   -0.4514    0.4976

   -0.4976   -0.4514    0.4514   -0.4976
    0.5486    0.4976   -0.4976    0.5486
```

```
1.0000    0
0         1.0000
```

The last two lines represent the 2×2 identity matrix.

1.12.5 Double-dot product

The double-dot product of two matrices, $\mathbf{T}$ and $\mathbf{S}$, is a scalar defined as

$$\mathbf{T} : \mathbf{S} \equiv \mathrm{trace}(\mathbf{T}^{\mathrm{T}} \cdot \mathbf{S}) = \mathrm{trace}(\mathbf{T} \cdot \mathbf{S}^{\mathrm{T}}), \tag{1.12.9}$$

where the superscript T denotes the matrix transpose. Substituting

$$\mathbf{T} = C_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{S} = D_{pq}\, \mathbf{e}_p \otimes \mathbf{e}_q, \tag{1.12.10}$$

we find that

$$\mathbf{T}^{\mathrm{T}} \cdot \mathbf{S} = C_{ji}\, D_{pq}\, (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_p \otimes \mathbf{e}_q) = C_{ji}\, D_{jq}\, (\mathbf{e}_i \otimes \mathbf{e}_q). \tag{1.12.11}$$

Since $\mathrm{trace}(\mathbf{e}_i \otimes \mathbf{e}_q) = \delta_{iq}$, we conclude that

$$\mathbf{T} : \mathbf{S} = C_{ji}\, D_{ji}, \tag{1.12.12}$$

where summation is implied over the repeated indices, i and j.

1.12.6 Universal Cartesian base

In the universal Cartesian base, $\mathbf{e}_i = \boldsymbol{\epsilon}_i$, where all entries of $\boldsymbol{\epsilon}_i$ are zero, except for the ith entry that is equal to 1. In three dimensions,

$$\boldsymbol{\epsilon}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.12.13}$$

The associated dyadic Cartesian base is described by nine matrices, where all elements of each matrix is zero, except for one element that is equal to unity. For example,

$$\mathbf{E}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.12.14}$$

In the universal Cartesian base, and only then, $\mathbf{C} = \mathbf{T}$. We may then identify the matrix $\mathbf{T}$ with the component matrix in the universal Cartesian frame.

Exercise

1.12.1 Derive the expressions shown in (1.12.7).

1.13 Change of Cartesian base

Consider an arbitrary Cartesian base with base vectors $\mathbf{e}_i$ for $i = 1, \ldots N$, and another arbitrary Cartesian base with base vectors $\widetilde{\mathbf{e}}_i$ for $i = 1, \ldots N$, as discussed in Section 1.8.

The components of an $N \times N$ matrix $\mathbf{T}$ in the corresponding dyadic bases are given by

$$C_{mn} = \mathbf{e}_m \cdot \mathbf{T} \cdot \mathbf{e}_n, \qquad \widetilde{C}_{mn} = \widetilde{\mathbf{e}}_m \cdot \mathbf{T} \cdot \widetilde{\mathbf{e}}_n \tag{1.13.1}$$

in terms of Cartesian unit vector projections. We will show that the two sets of components are related by a simple relation underlying the notion of a tensor.

1.13.1 Transformation matrix

The tilded base arrays are related to the untilded arrays by a linear transformation,

$$\widetilde{\mathbf{e}}_i = Q_{ij}\, \mathbf{e}_j, \tag{1.13.2}$$

where summation is implied over the repeated index j and the elements of a transformation matrix, $\mathbf{Q}$, are given by

$$Q_{ij} \equiv \widetilde{\mathbf{e}}_i \cdot \mathbf{e}_j, \tag{1.13.3}$$

as discussed in Section 1.8. Projecting (1.13.2) onto $\widetilde{\mathbf{e}}_m$, we obtain

$$\delta_{im} = Q_{ij}\, Q^{\mathrm{T}}_{jm}, \tag{1.13.4}$$

where m is a free index, δ_{im} is Kronecker's delta, and the superscript T denotes the matrix transpose. This relation shows that the transformation matrix is orthogonal,

$$\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}, \tag{1.13.5}$$

where the superscript -1 denotes the matrix inverse. Consequently,

$$\mathbf{e}_i = Q_{ji}\, \widetilde{\mathbf{e}}_j \tag{1.13.6}$$

The determinant of $\mathbf{Q}$ is equal to unity, as discussed in Section 1.8.

1.13.2 Matrix component transformation

To relate the two sets of components, we introduce the double expansion

$$\mathbf{T} = C_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j = \widetilde{C}_{ij}\, \widetilde{\mathbf{e}}_i \otimes \widetilde{\mathbf{e}}_j, \tag{1.13.7}$$

where summation is implied over the repeated indices, i and j. Projecting this expansion from the left onto $\widetilde{\mathbf{e}}_m$, where m is a free index, we obtain

$$\widetilde{\mathbf{e}}_m \cdot \mathbf{T} = C_{ij}\, (\widetilde{\mathbf{e}}_m \cdot \mathbf{e}_i)\, \mathbf{b}_j = \widetilde{C}_{ij}\, (\widetilde{\mathbf{e}}_m \cdot \widetilde{\mathbf{e}}_i)\, \widetilde{\mathbf{e}}_j, \tag{1.13.8}$$

which amounts to

$$\widetilde{\mathbf{e}}_m \cdot \mathbf{T} = C_{ij}\, Q_{mi}\, \mathbf{b}_j = \widetilde{C}_{mj}\, \widetilde{\mathbf{e}}_j. \tag{1.13.9}$$

Projecting (1.13.9) onto $\widetilde{\mathbf{e}}_n$, where n is another free index, we obtain

$$\widetilde{\mathbf{e}}_m \cdot \mathbf{T} \cdot \widetilde{\mathbf{e}}_n = C_{ij}\, Q_{mi}\, Q_{nj} = \widetilde{C}_{mn}. \tag{1.13.10}$$

We have found that

$$\widetilde{C}_{mn} = Q_{mi} C_{ij} Q^{\mathrm{T}}_{jn}, \qquad \widetilde{\mathbf{C}} = \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{1.13.11}$$

Working in a similar fashion, we obtain

$$C_{mn} = Q^{\mathrm{T}}_{mi} \widetilde{C}_{ij} Q_{jn}, \qquad \mathbf{C} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{C}} \cdot \mathbf{Q}, \tag{1.13.12}$$

which confirms further that the transformation matrix $\mathbf{Q}$ is orthogonal, that is, its inverse is equal to its transpose.

The components of $\mathbf{T}$ encapsulated in $\widetilde{\mathbf{C}}$ can be computed in two ways: (a) directly based on (1.13.1) or (b) indirectly in terms $\mathbf{C}$ based on (1.13.11). The results will be identical.

1.13.3 Confirmation by code

Confirmation is provided by the following Matlab code named *tensor*, located in directory TENSOR of TUNLIB, for a 2×2 matrix:

```
%---
% two bases, A and B, rotated by thA and thB
% in a plane
%---

thA = 0.034*pi;  % arbitrary
thB = 0.245*pi;  % arbitrary

eA1(1) = cos(thA); eA1(2) = sin(thA);
eA2(1) =-eA1(2);   eA2(2) = eA1(1);

eB1(1) = cos(thB); eB1(2) = sin(thB);
eB2(1) =-eB1(2);   eB2(2) = eB1(1);

%---
% transformation matrix
%---

Q(1,1) = eB1(1)*eA1(1) + eB1(2)*eA1(2);
Q(1,2) = eB1(1)*eA2(1) + eB1(2)*eA2(2);
Q(2,1) = eB2(1)*eA1(1) + eB2(2)*eA1(2);
Q(2,2) = eB2(1)*eA2(1) + eB2(2)*eA2(2);

%---
% confirm orthogonality
%---

[Q' inv(Q)]

det(Q)
```

```
%---
% tensor
%---

T = [ 1 2;
      3 4];
%---
% A components
%---

TA(1,1) = eA1*T*eA1'; TA(1,2) = eA1*T*eA2';
TA(2,1) = eA2*T*eA1'; TA(2,2) = eA2*T*eA2';

%---
% B components
%---

TB(1,1) = eB1*T*eB1'; TB(1,2) = eB1*T*eB2';
TB(2,1) = eB2*T*eB1'; TB(2,2) = eB2*T*eB2';

%---
% print
%---

[Q; TA; TB; Q*TA*Q']
```

Running the code generates the following output:

```
Q' and inv(Q)

    0.7882   -0.6154    0.7882   -0.6154
    0.6154    0.7882    0.6154    0.7882

det(Q)

    1.0000

Q   TA   TB   Q*TA*Q'

 0.7882 0.6154   1.5641 2.2612   4.9517 1.0778
```

```
                                  4.9517 1.0778
-0.6154 0.7882   3.2612 3.4359    2.0778 0.0483
                                  2.0778 0.0483
```

The fifth and sixth pairs of columns are identical to the seventh and eighth pairs of columns generated by a transformation.

1.13.4 Similarity transformation

Equations (1.13.11) and (1.13.12) express similarity transformations between two matrices, $\mathbf{C}$ and $\widetilde{\mathbf{C}}$. This means that the characteristic polynomial, and thus the eigenvalues, the determinant, and the trace of $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ are the same,

$$\mathrm{trace}(\mathbf{C}) = \mathrm{trace}(\widetilde{\mathbf{C}}), \qquad \det(\mathbf{C}) = \det(\widetilde{\mathbf{C}}), \qquad (1.13.13)$$

where the trance is sum of the diagonal elements.

1.13.5 Identity tensor

The identity matrix, $\mathbf{I}$, is distinguished by the property that $\mathbf{I} \cdot \mathbf{a} = \mathbf{a}$, for any vector, $\mathbf{a}$. We may readily confirm that $C_{mn} = \delta_{mn}$, and $\widetilde{C}_{mn} = \delta_{mn}$, which demonstrates that $\mathbf{I}$ is a tensor,

$$\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i = \widetilde{\mathbf{e}}_i \otimes \widetilde{\mathbf{e}}_i, \qquad (1.13.14)$$

where summation is implied over the repeated index, i.

1.13.6 Transformation in terms of coordinates

Using the expressions derived in Section 1.8.7 for the transformation matrix $\mathbf{Q}$, we find that the transformation rule (1.13.11) takes the form

$$\widetilde{C}_{mn} = \frac{\partial \widetilde{X}_m}{\partial X_i} \frac{\partial \widetilde{X}_n}{\partial X_j} C_{ij} = \frac{\partial X_i}{\partial \widetilde{X}_m} \frac{\partial \widetilde{X}_n}{\partial X_j} C_{ij} = \frac{\partial \widetilde{X}_m}{\partial X_i} \frac{\partial X_j}{\partial \widetilde{X}_n} C_{ij} = \frac{\partial X_i}{\partial \widetilde{X}_m} \frac{\partial X_j}{\partial \widetilde{X}_n} C_{ij}. \qquad (1.13.15)$$

The inverse transformation rule (1.13.12) takes the form

$$C_{mn} = \frac{\partial \widetilde{X}_i}{\partial X_m} \frac{\partial \widetilde{X}_j}{\partial X_n} \widetilde{C}_{ij} = \frac{\partial X_m}{\partial \widetilde{X}_i} \frac{\partial \widetilde{X}_j}{\partial X_n} \widetilde{C}_{ij}$$
$$= \frac{\partial \widetilde{X}_i}{\partial X_m} \frac{\partial X_n}{\partial \widetilde{X}_j} \widetilde{C}_{ij} = \frac{\partial X_m}{\partial \widetilde{X}_i} \frac{\partial X_n}{\partial \widetilde{X}_j} \widetilde{C}_{ij}. \quad (1.13.16)$$

Exercise

1.13.1 Compute the transformation matrix $\mathbf{Q}$ when $\mathbf{e}_i = \boldsymbol{\epsilon}_i$ is the universal Cartesian base.

1.14 Second-order tensors

Let the base arrays $\mathbf{e}_i$ define a Cartesian system in N-dimensional space, and the base arrays $\widetilde{\mathbf{e}}_i$ define another Cartesian system.

1.14.1 Two sets of measurements

Suppose that a measurement of a physical multi-scalar quantity, such as stress, is taken in the the first system and the recording is placed in a component matrix, $\mathbf{C}$. Suppose also that a measurement of the same physical quantity is taken in the second system and the recording is placed in another component matrix, $\widetilde{\mathbf{C}}$.

The two measurements must be related by (1.13.11) and (1.13.12), repeated below for convenience,

$$\widetilde{\mathbf{C}} = \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{C} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{C}} \cdot \mathbf{Q}, \qquad (1.14.1)$$

where the superscript T denotes the matrix transpose and $\mathbf{Q}$ is the pertinent orthogonal transformation matrix. If they are, the physical quantity possesses the attribute of a *second-order tensor*.

If a two-index entity does not obey the transformation test expressed by (1.13.11) and (1.13.12), then it is not a tensor and should not be used in physical frameworks.

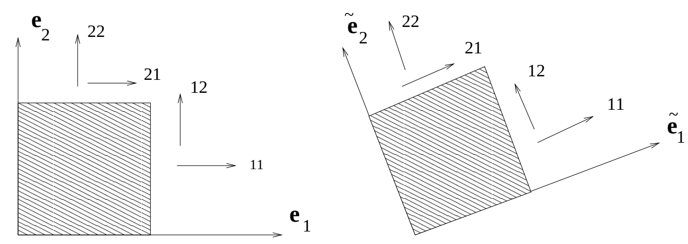

FIGURE 1.14.1 Stresses exerted at the surface of a small rectangular block of a fluid or solid in two coordinate systems. The stress components in the two systems are indicated by arrows.

1.14.2 Stress tensor

The terminology *tensor* can be traced to solid mechanics where materials are subjected to forces or deformation, and thereby develop internal stresses. The name *tensor* derives from the word *tension* (force per length), which is similar to *stress* (force per area)

The stress tensor in two dimensions encapsulates four stresses denoted by 11, 12, 21, and 22, as shown in Figure 1.14.1, where 11 and 22 are normal stresses and 12 and 21 are shear stresses. One set of measurements can be taken in the $\mathbf{e}_1\mathbf{e}_2$ system by a stress meter, and another set of measurements is taken in the $\widetilde{\mathbf{e}}_1\widetilde{\mathbf{e}}_2$ system by a rotated stress meter. In the absence of instrumentation error, the two sets of stress measurements must be related by (1.13.11) and (1.13.12).

1.14.3 Mohr transformation

In two dimensions, if the tilde system arises by rotating the untilded system around the origin by angle ϱ, then

$$\mathbf{Q} \equiv \begin{bmatrix} \cos\varrho & \sin\varrho \\ -\sin\varrho & \cos\varrho \end{bmatrix}, \tag{1.14.2}$$

according to the Mohr transformation. Applying (1.14.1), we find that

$$\widetilde{\mathbf{C}} = \begin{bmatrix} \cos\varrho & \sin\varrho \\ -\sin\varrho & \cos\varrho \end{bmatrix} \cdot \mathbf{C} \cdot \begin{bmatrix} \cos\varrho & -\sin\varrho \\ \sin\varrho & \cos\varrho \end{bmatrix}. \tag{1.14.3}$$

Carrying out the multiplications for $C_{12} = C_{21}$, we obtain

$$\begin{aligned} \widetilde{C}_{11} &= \cos^2\varrho\, C_{11} + \sin^2\varrho\, C_{22} + \sin 2\varrho\, C_{12}, \\ \widetilde{C}_{12} &= \sin\varrho\cos\varrho\,(C_{22} - C_{11}) + \cos 2\varrho\, C_{12}, \\ \widetilde{C}_{22} &= \sin^2\varrho\, C_{11} + \cos^2\varrho\, C_{22} - \sin 2\varrho\, C_{12}. \end{aligned} \tag{1.14.4}$$

Measurements of $\widetilde{C}_{ij}$ and C_{ij} must be consistent with these equations.

1.14.4 Tensor rule for the vector tensor product

An arbitrary vector, $\mathbf{v}$, may be expanded in two Cartesian bases in terms of its components in the two systems as

$$\mathbf{v} = c_j\, \mathbf{e}_j = \widetilde{c}_j\, \widetilde{\mathbf{e}}_j. \tag{1.14.5}$$

Another arbitrary vector, $\mathbf{u}$, may be expanded similarly as

$$\mathbf{u} = d_j\, \mathbf{e}_j = \widetilde{d}_j\, \widetilde{\mathbf{e}}_j. \tag{1.14.6}$$

The tensor product of these vectors is a matrix,

$$\mathbf{T} = \mathbf{u} \otimes \mathbf{v}, \tag{1.14.7}$$

with elements $T_{ij} = v_i u_j$, where v_i and u_j are the vector components in the universal Cartesian base, $\boldsymbol{\epsilon}_m$. We will confirm that $\mathbf{T}$ is a tensor.

Using the aforementioned expansions, we find that

$$\mathbf{T} = c_i\, d_j\, \mathbf{e}_i \otimes \mathbf{e}_j = \widetilde{c}_i\, \widetilde{d}_j\, \widetilde{\mathbf{e}}_i \otimes \widetilde{\mathbf{e}}_j. \tag{1.14.8}$$

The products

$$C_{ij} \equiv c_i\, d_j \tag{1.14.9}$$

are the components of $\mathbf{T}$ in the $\mathbf{E}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j$ base, and the products

$$\widetilde{C}_{ij} \equiv \widetilde{c}_i\, \widetilde{d}_j \tag{1.14.10}$$

are the components of $\mathbf{T}$ in the $\widetilde{\mathbf{E}}_{ij} \equiv \widetilde{\mathbf{e}}_i \otimes \widetilde{\mathbf{e}}_j$ base. We may write

$$\mathbf{T} = C_{ij}\, \mathbf{E}_{ij} = \widetilde{C}_{ij}\, \widetilde{\mathbf{E}}_{ij}, \tag{1.14.11}$$

where summation is implied over the repeated indices i and j.

Using the vector transformation rules (1.8.13), repeated below for convenience,

$$\widetilde{\mathbf{c}} = \mathbf{Q} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}, \\ \widetilde{\mathbf{d}} = \mathbf{Q} \cdot \mathbf{d}, \qquad \mathbf{d} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{d}}, \tag{1.14.12}$$

we find that

$$\widetilde{C}_{ij} \equiv \widetilde{c}_i\, \widetilde{d}_j \;= (Q_{ik} c_k)(Q_{jm} d_m). \tag{1.14.13}$$

Rearranging the products on the right-hand side, we obtain

$$\widetilde{C}_{ij} = Q_{ik}\, c_k\, d_m\, Q_{jm} = Q_{ik}\, C_{km}\, Q_{jm}. \tag{1.14.14}$$

In matrix notation, this equation takes the form

$$\widetilde{\mathbf{C}} = \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{T}}, \tag{1.14.15}$$

where the superscript T denotes the matrix transpose, thereby confirming by definition that the tensor product of two Cartesian vectors is a two-index or Cartesian tensor.

1.14.5 Second moment-of-inertia tensor

Consider a point-particle located at position $\mathbf{x}$ so that

$$\mathbf{x} = X_i\, \mathbf{e}_i, \tag{1.14.16}$$

where X_i are the particle coordinates in a Cartesian system with base vectors $\mathbf{e}_i$ whose origin coincides with that the universal base. The second-moment of inertia tensor is defined as $m\mathbf{J}$, where m is the particle mass,

$$\mathbf{J} \equiv |\mathbf{X}|^2\, \mathbf{I} - \mathbf{X} \otimes \mathbf{X}, \tag{1.14.17}$$

where $|\mathbf{X}|^2 = X_1^2 + X_2^2 + X_3^2$ is the distance from the origin and

$$\mathbf{X} \otimes \mathbf{X} = \begin{bmatrix} X_1^2 & X_1 X_2 & X_1 X_3 \\ X_2 X_1 & X_2^2 & X_2 X_3 \\ X_3 X_1 & X_3 X_2 & X_3^2 \end{bmatrix}, \tag{1.14.18}$$

In index notation,

$$J_{ij} = \delta_{ij}\,|\mathbf{X}^2| - X_i X_j. \qquad (1.14.19)$$

We will demonstrate that $\mathbf{J}$ is a tensor for any two Cartesian systems that share an origin, so that

$$\widetilde{X}_i = Q_{ij}\,X_j. \qquad (1.14.20)$$

To carry out the proof, we write

$$\widetilde{J}_{ij} = \delta_{ij}\,\widetilde{X}_p\widetilde{X}_p - \widetilde{X}_i\widetilde{X}_j \qquad (1.14.21)$$

and then

$$\widetilde{J}_{ij} = \delta_{ij}\,(Q_{pm}X_m)(Q_{pn}X_n) - (Q_{im}X_m)\,(Q_{jn}X_n). \qquad (1.14.22)$$

Rearranging, we obtain

$$\widetilde{J}_{ij} = \delta_{ij}\,Q^{\mathrm{T}}_{mp}Q_{pn}\,X_mX_n - Q_{im}(X_mX_n)\,Q_{jn}. \qquad (1.14.23)$$

Noting that $Q^{\mathrm{T}}_{mp}Q_{pn} = \delta_{mn}$, we obtain

$$\widetilde{J}_{ij} = \delta_{ij}\,X_qX_q - Q_{im}(X_mX_n)\,Q_{jn}. \qquad (1.14.24)$$

Finally, we use the orthogonality of the transformation matrix to write

$$\widetilde{J}_{ij} = Q_{im}\,(\delta_{ij}\,|\mathbf{X}|^2 - X_mX_n)\,Q_{jn}. \qquad (1.14.25)$$

which demonstrates the tensorial nature of $\mathbf{J}$.

In contrast, an entity $\mathbf{N}$ whose components in a certain Cartesian system are

$$N_{ij} = (-1)^{i+j}X_iX_j \qquad (1.14.26)$$

is not a tensor. Consequently, the matrix $\mathbf{N}$ cannot be used in a physical framework that passes the test of frame invariance.

1.14.6 Momentum tensor

Consider a point particle with mass m moving in space with velocity $\mathbf{u}$. We may resolve

$$\mathbf{u} = U_i\,\mathbf{e}_i, \tag{1.14.27}$$

where U_i are the velocity components in a coordinate system with base vectors $\mathbf{e}_i$. Working as previously in this section, we may confirm that the momentum tensor, $\mathbf{M}$, with components

$$M_{ij} = m\,U_i U_j \tag{1.14.28}$$

is a tensor. In fact, the tensorial nature of $\mathbf{M}$ becomes evident by noting that $\mathbf{M}$ is the tensor product of two identical vectors, $\mathbf{L} = \mathbf{u} \otimes \mathbf{u}$.

1.14.7 Proportionality tensors

If $\mathbf{v}$ is a vector and $\mathbf{u}$ is another vector arising from $\mathbf{u}$ by the transformation

$$\mathbf{u} = \mathbf{T} \cdot \mathbf{v}, \tag{1.14.29}$$

then the matrix $\mathbf{T}$ is a tensor. To demonstrate this, we write

$$\widetilde{\mathbf{u}} = \widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{v}} \tag{1.14.30}$$

for a tilded coordinate system, and then

$$\mathbf{A} \cdot \mathbf{u} = \widetilde{\mathbf{T}} \cdot \mathbf{A} \cdot \mathbf{v}, \tag{1.14.31}$$

which suggests that $\mathbf{T} = \mathbf{A}^{\mathrm{T}} \cdot \widetilde{\mathbf{T}} \cdot \mathbf{A}$, and thereby confirms that $\mathbf{T}$ is a tensor.

In fluid and solid mechanics, the vector $\mathbf{u}$ can be the traction, $\mathbf{v}$ can be a vector normal to a surface, and $\mathbf{T}$ can be the Cauchy stress tensor.

1.14.8 Velocity gradient tensor

The velocity-gradient tensor is the gradient of a velocity field, $\mathbf{u}$, defined as

$$\mathbf{L} \equiv \boldsymbol{\nabla}\mathbf{u}, \tag{1.14.32}$$

where $L_{ij} = \partial u_j / \partial x_i$. We may resolve $\mathbf{u} = U_i(\mathbf{x})\, \mathbf{e}_i$ and expand

$$\mathbf{L} = \frac{\partial U_j}{\partial X_i}\, \mathbf{e}_i \otimes \mathbf{e}_j, \tag{1.14.33}$$

where U_i are the velocity components and X_i are the coordinates in a Cartesian system with base vectors $\mathbf{e}_i$. Now using the chain rule, we write

$$\frac{\partial \widetilde{U}_j}{\partial \widetilde{X}_i} = \frac{\partial (Q_{jk}\, U_k)}{\partial \widetilde{X}_i} = Q_{jk}\, \frac{\partial U_k}{\partial X_p}\, \frac{\partial X_p}{\partial \widetilde{X}_i}. \tag{1.14.34}$$

Equation (1.8.22) identifies the last fraction on the right-hand side with Q_{ip}, yielding

$$\frac{\partial \widetilde{U}_j}{\partial \widetilde{X}_i} = Q_{ip}\, \frac{\partial U_k}{\partial X_p} Q^{\mathrm{T}}_{kj}, \tag{1.14.35}$$

in accordance with the rules of a second-order tensor.

Exercise

1.14.1 Show that the transpose of a tensor is also a tensor.

1.15 High-order tensors

High-order tensors are defined similarly to second-order tensors discussed previously in this chapter.

A third-order tensor admits the representation

$$\mathbf{T} = C_{ijk}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{1.15.1}$$

where C_{ijk} are the tensor components in a chosen Cartesian base, $\mathbf{e}_i$, given by

$$C_{ijk} = (\mathbf{e}_i \cdot \mathbf{T} \cdot \mathbf{e}_k) \cdot \mathbf{e}_j. \tag{1.15.2}$$

In double-dot product notation,

$$C_{ijk} = \mathbf{e}_i \cdot (\mathbf{T} : \mathbf{e}_j \otimes \mathbf{e}_k). \tag{1.15.3}$$

The double tensor products, $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ constitute a triadic base parametrized by three indices.

1.15.1 Universal Cartesian base

Referring to the universal Cartesian base, we set $\mathbf{e}_i = \boldsymbol{\epsilon}_i$ and find that all elements of the matrix $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$ are zero, except for the ijk element that is equal to unity. Consequently,

$$\mathbf{T} = T_{ijk}\, \boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k. \tag{1.15.4}$$

All elements of the three-index matrix $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$ are zero, expect for the ijk element that is equal to unity.

The transformation rule for a tilded Cartesian base is

$$\widetilde{C}_{mn\ell} = Q_{mi} Q_{nj} Q_{\ell k}\, C_{ijk}, \tag{1.15.5}$$

where $\mathbf{Q}$ is the relevant orthogonal transformation matrix and summation is implied over three repeated indices.

Exercise

1.15.1 Derive the transformation rule (1.15.5).

1.16 Alternating tensor

The three-index alternating tensor is defined by the expansion

$$\boldsymbol{\xi} = \epsilon_{ijk}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{1.16.1}$$

where ϵ_{ijk} is the Levi–Civita (LC) symbol introduced in Section 1.5. Since

$$\epsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k, \tag{1.16.2}$$

we may write

$$\boldsymbol{\xi} = \big((\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k \big)\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{1.16.3}$$

The expression inside the outer parentheses is the triple mixed scalar product.

Consequently, the components of $\boldsymbol{\xi}$ are equal to the elements of $\boldsymbol{\xi}$,

$$\xi_{ijk} = \epsilon_{ijk} \tag{1.16.4}$$

in any Cartesian base.

1.16.1 Base transformation

Any orthogonal matrix with unit determinant, $\mathbf{Q}$, satisfies the identity

$$\epsilon_{mn\ell} = Q_{mi} Q_{nj} Q_{\ell k}\, \epsilon_{ijk}. \tag{1.16.5}$$

Because of this identity, we may also write

$$\boldsymbol{\xi} = \epsilon_{ijk}\, \widetilde{\mathbf{e}}_i \otimes \widetilde{\mathbf{e}}_j \otimes \widetilde{\mathbf{e}}_k, \tag{1.16.6}$$

which confirms that the components of $\boldsymbol{\xi}$ remain constant in any Cartesian base.

The following Matlab code entitled *levciv1*, located in directory VECAR of TUNLIB, generates two Cartesian bases and confirms identity (1.16.5):

```
%----
% Cartesian base vectors A
% generated by random rotations
%----

th1 = rand*2.0*pi;
th2 = rand*2.0*pi;
th3 = rand*2.0*pi;

R1 = [1,0,0;
      0, cos(th1),sin(th1);
      0,-sin(th1),cos(th1)];

R2 = [cos(th2),0,-sin(th2);
```

```
      0,1,0;
      sin(th2),0,cos(th2)];

R3 = [ cos(th3),sin(th3),0;
      -sin(th3),cos(th3),0;
      0,0,1];

RA = R3*R2*R1;

for i=1:3
 eA1(i) = RA(i,1);
 eA2(i) = RA(i,2);
 eA3(i) = RA(i,3);
end

%----
% Cartesian base vectors B
% generated by random rotations
%----

th1 = rand*2.0*pi;
th2 = rand*2.0*pi;
th3 = rand*2.0*pi;

R1 = [1,0,0;
      0, cos(th1),sin(th1);
      0,-sin(th1),cos(th1)]

R2 = [cos(th2),0,-sin(th2);
      0,1,0;
      sin(th2),0,cos(th2)];

R3 = [ cos(th3),sin(th3),0;
      -sin(th3),cos(th3),0;
      0,0,1];

RB = R3*R2*R1;

for i=1:3
```

```
 eB1(i) = RB(i,1);
 eB2(i) = RB(i,2);
 eB3(i) = RB(i,3);
end

%---
% transformation matrix
%---

Q(1,1) = eB1(1)*eA1(1) + eB1(2)*eA1(2) + eB1(3)*eA1(3);
Q(1,2) = eB1(1)*eA2(1) + eB1(2)*eA2(2) + eB1(3)*eA2(3);
Q(1,3) = eB1(1)*eA3(1) + eB1(2)*eA3(2) + eB1(3)*eA3(3);

Q(2,1) = eB2(1)*eA1(1) + eB2(2)*eA1(2) + eB2(3)*eA1(3);
Q(2,2) = eB2(1)*eA2(1) + eB2(2)*eA2(2) + eB2(3)*eA2(3);
Q(2,3) = eB2(1)*eA3(1) + eB2(2)*eA3(2) + eB2(3)*eA3(3);

Q(3,1) = eB3(1)*eA1(1) + eB3(2)*eA1(2) + eB3(3)*eA1(3);
Q(3,2) = eB3(1)*eA2(1) + eB3(2)*eA2(2) + eB3(3)*eA2(3);
Q(3,3) = eB3(1)*eA3(1) + eB3(2)*eA3(2) + eB3(3)*eA3(3);

%---
% confirm transformation
%---

for m=1:3
 for n=1:3
  for l=1:3
   ssm = 0.0;
   for i=1:3
     for j=1:3
       for k=1:3
       ijk = (i-j)*(j-k)*(k-i)/2;
       inc = Q(m,i)*Q(n,j)*Q(l,k)*ijk;
       ssm = ssm + inc;
       end
     end
   end
   mnl = (m-n)*(n-l)*(l-m)/2; % levi-civita symbol
```

```
   if(abs(mnl)>0.0001)
   [mnl ssm]
   end
  end
 end
end
```

Running the code prints doublets of $(1, 1)$ or $(-1, -1)$.

1.16.2 Representation in terms of an arbitrary trio of vectors

Let $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, and $\mathbf{a}^{(3)}$ be three arbitrary vectors arranged at the three columns of a matrix, $\mathbf{A}$ so that first column of $\mathbf{A}$ is $\mathbf{a}^{(1)}$, the second column is $\mathbf{a}^{(2)}$, and the third column is $\mathbf{a}^{(3)}$. The alternating tensor admits the representation

$$\boldsymbol{\xi} = \epsilon_{ijk} \, \frac{1}{\det(\mathbf{A})} \, \mathbf{a}^{(i)} \otimes \mathbf{a}^{(j)} \otimes \mathbf{a}^{(k)}, \tag{1.16.7}$$

where

$$\det(\mathbf{A}) = \mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w}) \equiv [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}] \tag{1.16.8}$$

is the triple mixed product representing the volume of a parallelepiped whose sides are defined by $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, and $\mathbf{a}^{(3)}$. Cyclic permutation of the three vectors preserves the triple mixed product; non-cyclic permutation preserves the magnitude but changes the sign.

The following Matlab code entitled *levciv2*, located in directory VECAR of TUNLIB, confirms this expansion:

```
%---
% three arbitrary vectors
%---

a1 = [ 1.3, 4.2, 4.6];
a2 = [ 0.3, 2.1, 1.2];
a3 = [-0.3,-8.1, 0.1];

%---
```

```
% put vectors in a matrix
%---

A = [a1(1), a2(1), a3(1);
     a1(2), a2(2), a3(2);
     a1(3), a2(3), a3(3)];

%---
% sum
%---

for p=1:3
  for q=1:3
   for m=1:3
     xi(p,q,m) = 0.0;
     for i=1:3
      for j=1:3
        for k=1:3
         ijk = (i-j)*(j-k)*(k-i)/2;
         xi(p,q,m) = xi(p,q,m) + ijk*A(p,i)*A(q,j)*A(m,k);
       end
     end
    end
  end
 end
end

xi = xi/det(M);
xi
```

Running the code generates the following output, as instructed by the last line of the code:

```
xi(:,:,1) =
         0    0.0000   -0.0000
         0         0    1.0000
         0   -1.0000    0.0000

xi(:,:,2) =
   -0.0000   -0.0000   -1.0000
```

```
    0.0000         0         0
    1.0000         0   -0.0000

xi(:,:,3) =
         0    1.0000         0
   -1.0000         0   -0.0000
         0    0.0000         0
```

1.16.3 Cross product of two vectors

The cross or outer product of an ordered pair of vectors, $\mathbf{v}$ and $\mathbf{u}$, is another vector given by

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = \boldsymbol{\xi} : (\mathbf{v} \otimes \mathbf{u}), \tag{1.16.9}$$

where : denotes the double-dot product. To demonstrate this relation, we expand $\mathbf{v} = c_p \, \mathbf{e}_p$ and $\mathbf{u} = d_q \, \mathbf{e}_q$, and find that

$$\mathbf{w} = \epsilon_{ijk} \, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) : (c_p d_q \, \mathbf{e}^p \otimes \mathbf{e}^q), \tag{1.16.10}$$

which can be restated as

$$\mathbf{w} = \epsilon_{ijk} c_p d_q \, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) : (\mathbf{e}^p \otimes \mathbf{e}^q), \tag{1.16.11}$$

and then

$$\mathbf{w} = \epsilon_{ijk} \, c_j d_k \, \mathbf{e}_i = \epsilon_{ijk} \, c_j d_k \, \mathbf{e}_i, \tag{1.16.12}$$

which reproduces expression (1.5.17).

Exercise

1.16.1 Confirm that the matrices RA and RB in the code *levciv1* are orthogonal. What is the determinant of these matrices?

Chapter 2

Biorthogonal bases

Biorthogonal vector and tensor bases discussed in this chapter are constructed in terms of a *specified* collection of vectors and another *specific* collection of vectors that satisfy conjugate orthogonality conditions. The main advantage of using dual coordinates is that vector and tensor components in one base can be extracted efficiently using the conjugate base, and this considerably simplifies theoretical derivations numerical computation.

The study of biorthogonal bases serves as a natural introduction to the apparatus of contravariant and covariant coordinates whose base vectors generally vary with position in space.

2.1 Biorthogonal vector bases

Consider an arbitrary set of N linearly independent N-dimensional vectors, denoted by

$$\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N, \tag{2.1.1}$$

and introduce another set of N conjugate of dual linearly independent N-dimensional vectors, denoted by

$$\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^N, \tag{2.1.2}$$

with the property that

$$\mathbf{b}_i \cdot \mathbf{b}^j = 0 \quad \text{if } i \neq j. \tag{2.1.3}$$

The scalar self-product, $\mathbf{b}_i \cdot \mathbf{b}^i$ can be arbitrary. An example in a plane, $N = 2$, is shown in Figure 2.1.1.

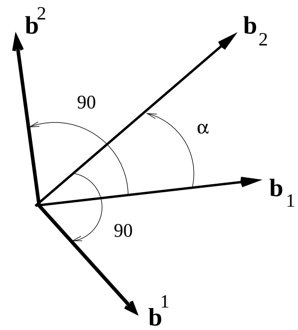

FIGURE 2.1.1 Illustration of conjugate vector bases, $(\mathbf{b}_1, \mathbf{b}_2)$ and $(\mathbf{b}^1, \mathbf{b}^2)$, in two-dimensional space. The base vectors satisfy the orthogonality property (2.1.3). When the angle α between $\mathbf{b}_1$ and $\mathbf{b}_2$ is equal to $\pi/2$, the two bases are parallel.

2.1.1 Covariant and contravariant bases

By convention, the first set of vectors, $\mathbf{b}_i$, is called *covariant*, and the second set of vectors, $\mathbf{b}^i$, is called *contravariant*. This terminology and the subscript/superscript notation stems from the theory of curvilinear coordinates, as discussed in subsequent chapters.

2.1.2 Biorthogonal projections

For future reference, we define the corresponding biorthogonal projections,

$$\omega^{(i)} \equiv \mathbf{b}_i \cdot \mathbf{b}^i \tag{2.1.4}$$

for $i = 1, \ldots, N$, where summation is *not* implied over the repeated index, i. In the following discussion, the superscript (i) of $\omega^{(i)}$ is *not* to be interpreted as an index; that is, the Einstein summation convention will not apply for this parenthesized superscript.

In the event that the two sets, $\mathbf{b}_i$ and $\mathbf{b}^i$ are *biorthonormal*, $\omega^{(i)} = 1$ for any i. Biorthonormal sets are employed in the apparatus of contravariant and covariant coordinates introduced in Chapters 4 and discussed in subsequent chapters.

The orthogonality condition (2.1.3), combined with the definition (2.1.4), allows us to write

$$\mathbf{b}_i \cdot \mathbf{b}^j = \omega^{(i)}\,\delta_{ij}, \tag{2.1.5}$$

where δ_{ij} is Kronecker's delta representing the identity matrix: $\delta_{ij} = 1$ if $i = j$, or 0 otherwise.

2.1.3 Matrices of base vectors

The components of the covariant set, $\mathbf{b}_i$, can be arranged at the *columns* of a matrix, $\mathbf{F}$, and those of the associated contravariant set, $\mathbf{b}^i$, can be arranged at the *columns* of another matrix, $\mathbf{\Phi}$,

$$\mathbf{F} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \qquad \mathbf{\Phi} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}^1 & \cdots & \mathbf{b}^N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}. \tag{2.1.6}$$

We may write

$$\mathbf{F} = \mathbf{b}_k \otimes \boldsymbol{\epsilon}_k, \qquad \mathbf{\Phi} = \mathbf{b}^k \otimes \boldsymbol{\epsilon}_k, \tag{2.1.7}$$

where summation is implied over the repeated index, k. By definition, all entries of the Cartesian array $\boldsymbol{\epsilon}_k$ are zero, except for the kth entry that is equal to 1, as discussed in Section 1.4.

2.1.4 Relation between $\mathbf{F}$ and $\mathbf{\Phi}$

Let $\boldsymbol{\omega}$ be a diagonal matrix whose mth diagonal element is equal to $\omega^{(m)}$. By construction,

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}} \cdot \mathbf{F} = \boldsymbol{\omega}, \tag{2.1.8}$$

where the superscript T denotes the matrix transpose. To confirm this equation, we write

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = (\boldsymbol{\epsilon}_k \otimes \mathbf{b}_k) \cdot (\mathbf{b}^m \otimes \boldsymbol{\epsilon}_m) = (\mathbf{b}_k \cdot \mathbf{b}^m)\,\boldsymbol{\epsilon}_k \otimes \boldsymbol{\epsilon}_m, \tag{2.1.9}$$

and thus

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \delta_{km}\,\omega^{(k)}\boldsymbol{\epsilon}_k \otimes \boldsymbol{\epsilon}_m = \omega^{(k)}\boldsymbol{\epsilon}_k \otimes \boldsymbol{\epsilon}_k, \tag{2.1.10}$$

where summation is implied over the repeated index, k. To complete the proof, we recall that $\boldsymbol{\epsilon}_k \otimes \boldsymbol{\epsilon}_k$ is the null matrix with *one* at the kth diagonal entry.

We have found that

$$\boldsymbol{\Phi} = \mathbf{F}^{-\mathrm{T}} \cdot \boldsymbol{\omega}, \qquad \mathbf{F} = \boldsymbol{\Phi}^{-\mathrm{T}} \cdot \boldsymbol{\omega}, \tag{2.1.11}$$

where the superscript $-\mathrm{T}$ denotes the inverse of the transpose, which is equal to the transpose of the inverse. If the two sets are biorthonormal, $\boldsymbol{\omega}$ is the identity matrix, $\mathbf{I}$, and $\boldsymbol{\Phi}$ is the inverse of $\mathbf{F}^{\mathrm{T}}$, and *vice versa*.

2.1.5 Biorthogonal construction

The two equations in (2.1.11) provide us with a practical method of computing contravariant from covariant matrices encapsulating base vectors, and *vice versa*, for given $\boldsymbol{\omega}$.

2.1.6 Cartesian base

If the base vectors constitute a Cartesian base,

$$\mathbf{b}^i = \mathbf{b}_i = \mathbf{e}_i \tag{2.1.12}$$

for $i = 1, \ldots, N$, the matrix $\boldsymbol{\omega}$ is the unit matrix, $\mathbf{F} = \boldsymbol{\Phi}$, and $\mathbf{F} = \mathbf{F}^{-\mathrm{T}}$, which shows that the matrices $\mathbf{F}$ and $\boldsymbol{\Phi}$ are orthogonal.

Exercise

2.1.1 Compute contravariant base vectors, $\mathbf{b}^i$, associated with the covariant base vectors $\mathbf{b}_1 = [2, 0]$ and $\mathbf{b}_2 = [3, 1]$.

2.2 Metric coefficients

It is useful to introduce two sets of coefficients,

$$b_{ij} \equiv \mathbf{b}_i \cdot \mathbf{b}_j, \qquad b^{ij} \equiv \mathbf{b}^i \cdot \mathbf{b}^j, \tag{2.2.1}$$

where b_{ij} are termed the *covariant metric coefficients* and b^{ij} are termed the *contravariant metric coefficients*. The former are arranged in a matrix $\mathbf{b}$ and the latter are arranged in a matrix $\boldsymbol{\beta}$ defined such that

$$[\mathbf{b}]_{ij} = b_{ij}, \qquad [\boldsymbol{\beta}]_{ij} = b^{ij}. \tag{2.2.2}$$

Thus, the ij element of the matrix $\mathbf{b}$ is equal to b_{ij} and the ij element of the matrix $\boldsymbol{\beta}$ is equal to b^{ij}.

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} b^{11} & b^{12} & b^{13} \\ b^{21} & b^{22} & b^{23} \\ b^{31} & b^{32} & b^{33} \end{bmatrix}. \tag{2.2.3}$$

By definition,

$$\mathbf{b} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \qquad \boldsymbol{\beta} = \boldsymbol{\Phi}^{\mathrm{T}} \cdot \boldsymbol{\Phi}. \tag{2.2.4}$$

2.2.1 Relation between dual metric coefficients

Using (2.1.11) to express $\boldsymbol{\Phi}$ in terms of $\mathbf{F}$ in the second expression of (2.2.4), we find that

$$\boldsymbol{\beta} = \boldsymbol{\omega} \cdot \mathbf{F}^{-1} \cdot \mathbf{F}^{-\mathrm{T}} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F})^{-1} \cdot \boldsymbol{\omega}. \tag{2.2.5}$$

Simplifying, we find that

$$\boldsymbol{\beta} = \boldsymbol{\omega} \cdot \mathbf{b}^{-1} \cdot \boldsymbol{\omega}, \qquad \mathbf{b} = \boldsymbol{\omega} \cdot \boldsymbol{\beta}^{-1} \cdot \boldsymbol{\omega}. \tag{2.2.6}$$

The first of these equations can be restated as

$$\boldsymbol{\omega}^{-1} \cdot \boldsymbol{\beta} = (\boldsymbol{\omega}^{-1} \cdot \mathbf{b})^{-1}, \tag{2.2.7}$$

which shows that

$$\boldsymbol{\omega}^{-1} \cdot \mathbf{b} \cdot \boldsymbol{\omega}^{-1} \cdot \boldsymbol{\beta} = \mathbf{I}, \qquad \boldsymbol{\omega}^{-1} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\omega}^{-1} \cdot \mathbf{b} = \mathbf{I}. \tag{2.2.8}$$

If $\boldsymbol{\omega}$ is the identity matrix, and only then, $\mathbf{b}$ is the inverse of $\boldsymbol{\beta}$. In index notation, the first equation in (2.2.8) takes the form

$$\frac{1}{\omega^{(j)}}\, b_{ij}\, b^{jk} = \omega^{(i)}\, \delta_{ik}, \tag{2.2.9}$$

where summation is implied over the repeated index, j.

2.2.2 Jacobians

The Jacobian of the covariant and contravariant bases are defined as

$$\mathcal{J}_\circ = \det(\mathbf{F}), \qquad \mathcal{J}^\circ = \det(\mathbf{\Phi}). \tag{2.2.10}$$

Taking the determinant of both sides of the equation $\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \boldsymbol{\omega}$, and recalling that the determinant of a matrix is equal to the determinant of the transpose, we find that

$$\mathcal{J}_\circ \mathcal{J}^\circ = \det(\boldsymbol{\omega}), \tag{2.2.11}$$

where

$$\det(\boldsymbol{\omega}) = \prod_{i=1}^{N} \omega^{(i)}. \tag{2.2.12}$$

Now we recall the definitions $\mathbf{b} \equiv \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ and $\boldsymbol{\beta} \equiv \mathbf{\Phi}^{\mathrm{T}} \cdot \mathbf{\Phi}$, and find that

$$\det(\mathbf{b}) = \mathcal{J}_\circ^2, \qquad \det(\boldsymbol{\beta}) = \mathcal{J}^{\circ 2}. \tag{2.2.13}$$

Using these expressions, we find that

$$\frac{\mathcal{J}^\circ}{\mathcal{J}_\circ} = \frac{\det(\boldsymbol{\omega})}{\det(\mathbf{b})}, \qquad \frac{\mathcal{J}_\circ}{\mathcal{J}^\circ} = \frac{\det(\boldsymbol{\omega})}{\det(\boldsymbol{\beta})}. \tag{2.2.14}$$

2.2.3 Scaled metric coefficients

We have found that

$$\widehat{\mathbf{b}} \cdot \widehat{\boldsymbol{\beta}} = \mathbf{I}, \qquad \widehat{\mathbf{b}}^{-1} = \widehat{\boldsymbol{\beta}}, \qquad \widehat{\boldsymbol{\beta}}^{-1} = \widehat{\mathbf{b}}, \tag{2.2.15}$$

where

$$\widehat{\mathbf{b}} \equiv \boldsymbol{\omega}^{-1} \cdot \mathbf{b} \qquad \widehat{\boldsymbol{\beta}} \equiv \boldsymbol{\omega}^{-1} \cdot \boldsymbol{\beta} \tag{2.2.16}$$

are scaled matrices defined such that

$$\widehat{b}_{ij} \equiv \frac{1}{\omega^{(i)}}\, b_{ij}, \qquad \widehat{\beta}_{ij} \equiv \frac{1}{\omega^{(i)}}\, \beta_{ij} = \frac{1}{\omega^{(i)}}\, b^{ij}. \tag{2.2.17}$$

Note that b_{ij} is equal to b_{ji} and β_{ij} is equal to β_{ji}, but $\widehat{b}_{ij}$ is not necessarily equal to $\widehat{b}_{ji}$ and $\widehat{\beta}^{ij}$ is not necessarily equal to $\widehat{\beta}^{ji}$.

2.2.4 Contravariant from covariant base vectors

Each contravariant base vector can be expressed in terms of the covariant base vectors in a linear combination involving the matrix $\widehat{\beta}_{ij}$ defined in (2.2.17),

$$\mathbf{b}^i = \widehat{\beta}_{ji}\,\mathbf{b}_j, \tag{2.2.18}$$

where summation is implied over the repeated index, j. To prove this assertion, we project this formula onto $\mathbf{b}^m$, where m is a free index, and obtain

$$b^{im} = \widehat{\beta}_{ji}\,\mathbf{b}_j \cdot \mathbf{b}^m = \widehat{\beta}_{ji}\,\delta_{jm}\,\omega^{(m)} = \widehat{\beta}_{mi}\,\omega^{(m)}, \tag{2.2.19}$$

which reproduces the definition of $\widehat{\beta}_{mj}$. Unfortunately, relation (2.2.18) is circular: to compute $\mathbf{b}^i$ in terms of $\mathbf{b}_j$, we need all of β_{ij} and $\omega^{(j)}$, which are defined in terms of the set $\mathbf{b}^i$.

2.2.5 Covariant from contravariant base vectors

Conversely, each covariant vector, $\mathbf{b}_i$, can be expressed in terms of the contravariant vectors in a linear combination involving the matrix $\widehat{b}_{ij}$ defined in (2.2.17),

$$\mathbf{b}_i = \widehat{b}_{ji}\,\mathbf{b}^j, \tag{2.2.20}$$

where summation is implied over the repeated index, j. To prove this assertion, we project this equation onto $\mathbf{b}_m$, where m is a free index, we obtain

$$b_{im} = \widehat{b}_{ji}\,\mathbf{b}^j \cdot \mathbf{b}_m = \widehat{b}_{ji}\,\delta_{jm}\,\omega^{(m)} = \widehat{b}_{mi}\,\omega^{(m)}, \tag{2.2.21}$$

which reproduces the definition of $\widehat{b}_{jm}$. Unfortunately, relation (2.2.20) is circular: to compute $\mathbf{b}_i$ from $\mathbf{b}^j$, we need all of b_{ij} and $\omega^{(j)}$, which are defined in terms of the set $\mathbf{b}_j$.

Substituting (2.2.18) into (2.2.20), we find that

$$\mathbf{b}_j = \widehat{\beta}_{mi}\,\widehat{b}_{ij}\,\mathbf{b}_m, \tag{2.2.22}$$

where summation is implied over the repeated indices, i and m. This equation confirms that the metric coefficients satisfy property (2.2.15), that is, $\widehat{\beta}_{mi}\,\widehat{b}_{ij} = \delta_{mj}$.

2.2.6 Confirmation by code

The following Matlab code named *bio*, located in directory BIO of TUNLIB, performs the following functions: (a) it defines covariant base vectors in two dimensions, (b) it computes the contravariant from the covariant base vectors by rotation, around the origin, as shown in Figure 2.1.1, and (c) it confirms equations (2.2.15):

```
%---
% covariant base vectors
% bcov1 and bcov2
%---

thbcov1 = 0.034*pi;    % arbitrary
thbcov2 = 0.334*pi;    % arbitrary

lb1 = 1.4;  % arbitrary
lb2 = 1.8;  % arbitrary

bcov1(1) = lb1*cos(thbcov1); bcov1(2) = lb1*sin(thcovb1);
bcov2(1) = lb2*cos(thbcov2); bcov2(2) = lb2*sin(thcovb2);

%---
% contravariant base vectors
% bcon1 and bcon2
%---

thbcon1 = thbcov2 - 0.5*pi;
thbcon2 = thbcov1 + 0.5*pi;

lc1 = 2.4; % arbitrary
lc2 = 1.2; % arbitrary

bcon1(1) = lc1*cos(thbcon1); bcon1(2) = lc1*sin(thbcon1);
bcon2(1) = lc2*cos(thbcon2); bcon2(2) = lc2*sin(thbcon2);
```

```
%---
% projections
%---

omg(1) = bcov1*bcon1'; omg(2) = bcov2*bcon2';

%---
% compute covmet (b)
%---

covmet(1,1) = bcov1*bcov1'; covmet(1,2) = bcov1*bcov2';
covmet(2,1) = bcov2*bcov1'; covmet(2,2) = bcov2*bcov2';

%---
% compute conmet (beta)
%---

conmet(1,1) = bcon1*bcon1'; conmet(1,2) = bcon1*bcon2';
conmet(2,1) = bcon2*bcon1'; conmet(2,2) = bcon2*bcon2';

%---
% scaled matrices (hat)
%---

for i=1:2
 for j=1:2
   hatcovmet(i,j) = covmet(i,j)/omg(i);
   hatconmet(i,j) = conmet(i,j)/omg(i);
 end
end

%---
% confirm orthogonality
%---

[inv(hatcovmet) hatconmet]

%---
% confirm base vectors
```

```
%---

bcon1_conf = hatconmet(1,1)*bcov1 + hatconmet(2,1)*bcov2;
bcon2_conf = hatconmet(1,2)*bcov1 + hatconmet(2,2)*bcov2;

bcov1_conf = hatcovmet(1,1)*bcon1 + hatcovmet(2,1)*bcon2;
bcov2_conf = hatcovmet(1,2)*bcon1 + hatcovmet(2,2)*bcon2;

%---
% print
%---

[bcov1, bcov1_conf;
 bcov2, bcov2_conf;
 bcon1, bcon1_conf;
 bcon2, bcon2_conf]
```

Running the code generates the following output:

```
    2.1190   -0.6228        2.1190   -0.6228
   -0.9687    0.8240       -0.9687    0.8240

    1.3920    0.1493    1.3920    0.1493
    0.8967    1.5607    0.8967    1.5607
    2.0810   -1.1956    2.0810   -1.1956
   -0.1279    1.1932   -0.1279    1.1932
```

The first pair of columns is identical to the second pair, as required.

2.2.7 Cartesian base

If the base vectors constitute a Cartesian base,

$$\mathbf{b}^i = \mathbf{b}_i = \mathbf{e}_i \tag{2.2.23}$$

for $i = 1, \ldots, N$, the matrix $\boldsymbol{\omega}$ is the unit matrix, the matrix $\mathbf{F} = \boldsymbol{\Phi}$ is orthogonal, and $\mathbf{b} = \boldsymbol{\beta}$ are both equal to the identity matrix.

Exercise

2.2.1 Run the code *bio* for an orthogonal but non-Cartesian pair of covariant base vectors and discuss the results.

2.3 Vector components

An arbitrary vector, $\mathbf{v}$, can be expanded in two ways in terms of covariant or contravariant base vectors,

$$\mathbf{v} = v^i \, \mathbf{b}_i = v_i \, \mathbf{b}^i, \tag{2.3.1}$$

where v^i are contravariant vector components, v_i are covariant vector components, and summation is implied over the repeated index, i.

Note that the contravariant components, v^i, refer to the covariant base, $\mathbf{b}_i$, whereas the covariant components, v_i, refer to the contravariant base, $\mathbf{b}^i$.

2.3.1 Notational inconsistency

An unfortunate notational inconsistency has been introduced to conform with standard convention. The ith covariant vector component is denoted as v_i, but he ith element of the Cartesian vector $\mathbf{v}$ is also denoted as v_i. For clarity, the Cartesian components could be denoted with a Greek index, such as v_α for $\alpha = 1, \ldots, N$.

2.3.2 Computation of vector components

Projecting the double expansion (2.3.1) onto $\mathbf{b}_n$, where n is a free index, we obtain

$$\mathbf{v} \cdot \mathbf{b}_n = v^i \, \mathbf{b}_i \cdot \mathbf{b}_n = v_i \, \mathbf{b}^i \cdot \mathbf{b}_n, \tag{2.3.2}$$

yielding

$$\mathbf{v} \cdot \mathbf{b}_n = v^i \, b_{in} = v_n \, \omega^{(n)}. \tag{2.3.3}$$

Rearranging, we obtain an expression for the covariant components in terms of the contravariant vector components,

$$v_n = \frac{1}{\omega^{(n)}} \, \mathbf{v} \cdot \mathbf{b}_n = \frac{1}{\omega^{(n)}} \, b_{in} \, v^i = \widehat{b}_{ni} \, v^i, \tag{2.3.4}$$

where summation is implied over the repeated index, i.

Now projecting the double expansion (2.3.1) onto $\mathbf{b}^n$, where n is a free index, we obtain

$$\mathbf{v} \cdot \mathbf{b}^n = v^i \, \mathbf{b}_i \cdot \mathbf{b}^n = v_i \, \mathbf{b}^i \cdot \mathbf{b}^n, \tag{2.3.5}$$

yielding

$$\mathbf{v} \cdot \mathbf{b}^n = v^n \, \omega^{(n)} = v_i \, b^{in}. \tag{2.3.6}$$

Rearranging, we obtain an expression for the contravariant in terms of the covariant vector components,

$$v^n = \frac{1}{\omega^{(n)}} \, \mathbf{v} \cdot \mathbf{b}^n = \frac{1}{\omega^{(n)}} \, b^{in} \, v_i = \widehat{\beta}_{ni} \, v_i, \tag{2.3.7}$$

where summation is implied over the repeated index, i.

2.3.3 Raising and lowering indices

Formulas (2.3.4) and (2.3.7) provide us with rules for lowering or raising the indices, that is, for computing contravariant from covariant components and *vice versa*,

$$v_n = \widehat{b}_{ni} \, v^i, \qquad v^n = \widehat{\beta}_{ni} \, v_i. \tag{2.3.8}$$

Note the similarity and differences between this pair of equations and the pair of equations (2.2.20) and (2.2.18) repeated below for convenience,

$$\mathbf{b}_n = \widehat{b}_{in} \, \mathbf{b}^i \qquad \mathbf{b}^n = \widehat{\beta}_{in} \, \mathbf{b}_i. \tag{2.3.9}$$

We recall that the scaled matrices $\widehat{\mathbf{b}}$ and $\widehat{\boldsymbol{\beta}}$ are not necessarily symmetric.

2.3.4 Confirmation by code

The following code continuing code *bio* listed in Section 2.2, computes the vector components and confirms the conversion formulas:

```
%---
% arbitrary vector
%---

v = [-2.9 1.3];

v1_con = v*bcon1'/omg(1);
v2_con = v*bcon2'/omg(2);

v1_cov = v*bcov1'/omg(1);
v2_cov = v*bcov2'/omg(2);

v1_cov_test = hatcovmet(1,1)*v1_con+hatcovmet(1,2)*v2_con;
v2_cov_test = hatcovmet(2,1)*v1_con+hatcovmet(2,2)*v2_con;

v1_con_test = hatconmet(1,1)*v1_cov+hatconmet(1,2)*v2_cov;
v2_con_test = hatconmet(2,1)*v1_cov+hatconmet(2,2)*v2_cov;

[v1_cov v1_cov_test;
 v2_cov v2_cov_test;
 v1_con v1_con_test;
 v2_con v2_con_test]
```

Running the code generates the following output:

```
   -1.4137   -1.4137
   -0.3271   -0.3271
   -2.7919   -2.7919
    1.0999    1.0999
```

The first pair of columns is identical to the second pair, as required.

2.3.5 *Inner product of two vectors*

The inner product of two vectors, $\mathbf{v}$ and $\mathbf{u}$, is a scalar given by

$$\mathbf{v} \cdot \mathbf{u} = (v^i \mathbf{b}_i) \cdot (u_j \mathbf{b}^j) = v^i u_j \, \mathbf{b}_i \cdot \mathbf{b}^j, \tag{2.3.10}$$

where summation is implied over the repeated indices, i and j. Simplifying, we obtain

$$\mathbf{v} \cdot \mathbf{u} = v^i u_j \, \delta_{ij} \, \omega^{(i)} = v^i u_i \, \omega^{(i)}, \tag{2.3.11}$$

where summation is implied over the repeated index i. Working in a similar fashion, we obtain

$$\mathbf{v} \cdot \mathbf{u} = v_i \, u^i \, \omega^{(i)}, \tag{2.3.12}$$

where summation is implied over the repeated index, i.

We conclude that

$$\mathbf{v} \cdot \mathbf{u} = v^i u_i \, \omega^{(i)} = v_i \, u^i \, \omega^{(i)}, \tag{2.3.13}$$

involving corresponding pairs of contravariant and covariant components.

Now using the rules for raising and lowering indices, we find that

$$v_i \, u^i \, \omega^{(i)} = (\frac{1}{\omega^{(i)}} \, b_{im} v^m) \, (\frac{1}{\omega^{(i)}} \, b^{in} v_n) \, \omega^{(i)} \tag{2.3.14}$$

and then

$$v_i \, u^i \, \omega^{(i)} = (\frac{1}{\omega^{(i)}} \, b_{mi} \, b^{in}) v^m v_n. \tag{2.3.15}$$

According to (2.2.9), the expression inside the parentheses on the right-hand side is equal to $\omega^{(n)} \, \delta_{mn}$, thereby reconciling the two expressions given in (2.3.13).

2.3.6 Summary

A summary of definitions and properties pertaining to biorthogonal bases and vector components is given in Tables 2.3.1 and 2.3.2.

Exercise

2.3.1 Confirm that the expression inside the parentheses on the right-hand side of (2.3.15) is equal to $\omega^{(n)} \, \delta_{mn}$,

$\mathbf{b}_i$	covariant base vectors
$\mathbf{b}^i$	contravariant base vectors
$\omega^{(i)} \equiv \mathbf{b}_i \cdot \mathbf{b}^i$	diagonal projections
$\mathbf{b}_i \cdot \mathbf{b}^j = \delta_{ij}\, \omega^{(i)}$	biorthogonality condition
$\mathbf{F} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$	matrix of covariant base vectors
$\mathbf{\Phi} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}^1 & \cdots & \mathbf{b}^N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$	matrix of contravariant base vectors
$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}} \cdot \mathbf{F} = \boldsymbol{\omega}$	biorthogonality condition
$b_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j$	covariant metric coefficients
$\beta_{ij} = b^{ij} = \mathbf{b}^i \cdot \mathbf{b}^j$	contravariant metric coefficients
$\widehat{b}_{ij} = \dfrac{1}{\omega^{(i)}}\, b_{ij}$	scaled covariant metric coefficients
$\widehat{\beta}_{ij} = \dfrac{1}{\omega^{(i)}}\, \beta_{ij} = \dfrac{1}{\omega^{(i)}}\, b^{ij}$	scaled contravariant metric coefficients
$\mathbf{b} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$	covariant metric coefficients matrix
$\boldsymbol{\beta} = \mathbf{\Phi}^{\mathrm{T}} \cdot \mathbf{\Phi}$	contravariant metric coefficients matrix
$\widehat{\mathbf{b}}^{-1} = \widehat{\boldsymbol{\beta}}$	biorthogonality condition

TABLE 2.3.1 Definitions, properties, and miscellaneous relations pertaining to dual biorthogonal bases.

$\mathbf{b}_i = \widehat{b}_{ji}\,\mathbf{b}^j$	covariant from contravariant base vectors
$\mathbf{b}^i = \widehat{\beta}_{ji}\,\mathbf{b}_j$	contravariant from covariant base vectors
$v_i = \widehat{b}_{ij}\,v^j$	covariant from contravariant vector components
$v^i = \widehat{\beta}_{ij}\,v_j$	contravariant from covariant vector components

TABLE 2.3.2 Definitions, properties, and miscellaneous relations pertaining to dual biorthogonal bases.

2.4 Three dimensions

The two equations in (2.1.11) provide us with a practical method of computing contravariant from covariant matrices encapsulating base vectors and *vice versa*, in terms of a matrix inverse. In two dimensions, contravariant base vectors can be computed from covariant base vectors, and *vice versa*, by $\pm 90^\circ$ planar rotations. Explicit construction formulas are available in three dimensions.

2.4.1 Contravariant from covariant vectors in three dimensions

In three dimensions, contravariant base vectors can be computed from the covariant base vectors using the formula

$$\mathbf{b}^i = \tfrac{1}{2}\,\frac{1}{\mathcal{J}_\circ}\,\omega^{(i)}\,\epsilon_{ijk}\,\mathbf{b}_j \times \mathbf{b}_k, \tag{2.4.1}$$

where ϵ_{ijk} is the Levi–Civita symbol, summation is implied over the repeated indices, j and k, and

$$\mathcal{J}_\circ \equiv \mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3) = \det(\mathbf{F}) \tag{2.4.2}$$

is the assumed positive volume of a parallelepiped defined by $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Explicitly,

$$\mathbf{b}^1 = \frac{1}{\mathcal{J}_\circ}\,\omega^{(1)}\,\mathbf{b}_2 \times \mathbf{b}_3,$$
$$\mathbf{b}^2 = \frac{1}{\mathcal{J}_\circ}\,\omega^{(2)}\,\mathbf{b}_3 \times \mathbf{b}_1, \tag{2.4.3}$$
$$\mathbf{b}^3 = \frac{1}{\mathcal{J}_\circ}\,\omega^{(3)}\,\mathbf{b}_1 \times \mathbf{b}_2.$$

Based on these representations, we find that

$$\mathbf{b}_i \times \mathbf{b}_j = \mathcal{J}_\circ\,\frac{1}{\omega^{(k)}}\,\epsilon_{ijk}\,\mathbf{b}^k, \tag{2.4.4}$$

which confirms that $\mathbf{b}^k$ is perpendicular to $\mathbf{b}_i$ and $\mathbf{b}_j$ for $k \neq i, j$.

2.4.2 Covariant from contravariant vectors in three dimensions

Conversely, the contravariant base vectors can be computed from the covariant base vectors using the expression

$$\mathbf{b}_i = \tfrac{1}{2}\,\frac{1}{\mathcal{J}^\circ}\,\omega^{(i)}\,\epsilon_{ijk}\,\mathbf{b}^j \times \mathbf{b}^k, \tag{2.4.5}$$

where summation is implied over the repeated indices, j and k,

$$\mathcal{J}^\circ \equiv \mathbf{b}^1 \cdot (\mathbf{b}^2 \times \mathbf{b}^3) = \det(\mathbf{\Phi}) \tag{2.4.6}$$

is the assumed positive volume of a parallelepiped defined by $\mathbf{b}^1$, $\mathbf{b}^2$, and $\mathbf{b}^3$. Explicitly,

$$\mathbf{b}_1 = \frac{1}{\mathcal{J}^\circ}\,\omega^{(1)}\,\mathbf{b}^2 \times \mathbf{b}^3,$$
$$\mathbf{b}_2 = \frac{1}{\mathcal{J}^\circ}\,\omega^{(2)}\,\mathbf{b}^3 \times \mathbf{b}^1, \tag{2.4.7}$$
$$\mathbf{b}_3 = \frac{1}{\mathcal{J}^\circ}\,\omega^{(3)}\,\mathbf{b}^1 \times \mathbf{b}^2.$$

Based on these representations, we find that

$$\mathbf{b}^i \times \mathbf{b}^j = \mathcal{J}^\circ\,\frac{1}{\omega^{(k)}}\,\epsilon_{ijk}\,\mathbf{b}_k, \tag{2.4.8}$$

which confirms that $\mathbf{b}_k$ is perpendicular to $\mathbf{b}^i$ and $\mathbf{b}^j$ for $k \neq i, j$.

2.4.3 Cross product

The cross or outer ($\times$) product of two three-dimensional vectors, $\mathbf{v}$ and $\mathbf{u}$, is another vector given by

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = (v^i \mathbf{b}_i) \times (u^j \mathbf{b}_j) = (v_i \mathbf{b}^i) \times (u_j \mathbf{b}^j), \tag{2.4.9}$$

where summation is implied over the repeated indices, i and j. Distributing the multiplications, we find that

$$\mathbf{w} = v^i u^j \, \mathbf{b}_i \times \mathbf{b}_j = v_i u_j \, \mathbf{b}^i \times \mathbf{b}^j. \tag{2.4.10}$$

Now recalling equations (2.4.4) and (2.4.8), we find that

$$\mathbf{w} = \mathcal{J}^\circ \, \epsilon_{kij} \, v_i u_j \, \frac{1}{\omega^{(k)}} \, \mathbf{b}_k = \mathcal{J}_\circ \, \epsilon_{kij} \, v^i u^j \, \frac{1}{\omega^{(k)}} \, \mathbf{b}^k, \tag{2.4.11}$$

where summation is implied over the repeated indices, i, j, and k.

We have found that the contravariant and covariant components of $\mathbf{w}$ are given by

$$w^k = \mathcal{J}^\circ \, \frac{1}{\omega^{(k)}} \, \epsilon_{kij} \, v_i u_j, \qquad w_k = \mathcal{J}_\circ \, \frac{1}{\omega^{(k)}} \, \epsilon_{kij} \, v^i u^j. \tag{2.4.12}$$

Corresponding pairs of vector components are involved in these expressions.

2.4.4 An identity

Now using the rules for lowering indices, we find that

$$w_n = \frac{1}{\omega^{(n)}} \, b_{kn} \, w^k = \mathcal{J}^\circ \, \frac{1}{\omega^{(n)}} \frac{1}{\omega^{(k)}} \epsilon_{ijk} \, b_{kn} \, v_i u_j, \tag{2.4.13}$$

and then

$$w_n = \mathcal{J}^\circ \, \frac{1}{\omega^{(n)}} \frac{1}{\omega^{(k)}} \epsilon_{ijk} \, b_{kn} \frac{1}{\omega^{(i)}} \, b_{pi} \, v^p \, \frac{1}{\omega^{(j)}} \, b_{qj} u^q, \tag{2.4.14}$$

which can be rearranged into

$$w_n = \mathcal{J}^\circ \, \frac{1}{\omega^{(k)}} \frac{1}{\omega^{(i)}} \frac{1}{\omega^{(j)}} \epsilon_{ijk} \, b_{kn} \, b_{pi} \, b_{qj} \, v^p \, u^q \, \frac{1}{\omega^{(n)}}. \tag{2.4.15}$$

Comparing this equation with the second equation in (2.4.12), we derive the identity

$$\mathcal{J}_\circ \, \epsilon_{pqn} = \mathcal{J}^\circ \, \frac{1}{\omega^{(k)}} \frac{1}{\omega^{(i)}} \frac{1}{\omega^{(j)}} \epsilon_{ijk} \, b_{kn} \, b_{pi} \, b_{qj}. \tag{2.4.16}$$

which can be rearranged as

$$\epsilon_{pqn} = \frac{1}{\det(\mathbf{b})} \, \epsilon_{ijk} b_{ip} \, b_{jq} \; b_{kn}. \tag{2.4.17}$$

In fact, this identity is satisfied for any symmetric matrix, $\mathbf{b}$.

Exercise

2.4.1 Confirm by numerical computation identity (2.4.17) for a symmetric matrix $\mathbf{b}$ of your choice.

2.5 Biorthogonal dyadic tensor bases

The covariant and contravariant base vectors can be used to compose tensor bases in four combinations.

2.5.1 Covariant–contravariant base

A matrix base can be formulated in terms of a covariant–contravariant biorthogonal dyadic product as

$$\mathbf{B}_i^{\circ j} = \mathbf{b}_i \otimes \mathbf{b}^j, \tag{2.5.1}$$

where the circular symbol ($\circ$) serves as a single-space holder that can be interpreted as empty space. This notation means that

$$[\mathbf{B}_i^{\circ j}]_{k\ell} = [\mathbf{b}_i]_k \times [\mathbf{b}^j]_\ell, \tag{2.5.2}$$

where $[\mathbf{B}_i^{\circ j}]_{k\ell}$ is the $k\ell$ component of $\mathbf{B}_i^{\circ j}$, $[\mathbf{b}_i]_k$ is the kth component of $\mathbf{b}_i$ $[\mathbf{b}^j]_\ell$ is the ℓth component of $\mathbf{b}_j$, and $\times$ denotes regular scalar multiplication.

2.5.2 *Tensor expansion*

An arbitrary tensor, $\mathbf{T}$, can be expanded as

$$\mathbf{T} = T^i_{\circ j} \mathbf{B}^{\circ j}_i = T^i_{\circ j}\, \mathbf{b}_i \otimes \mathbf{b}^j, \tag{2.5.3}$$

where $T^i_{\circ j}$ are the matrix components associated with the covariant–contravariant dyadic basis, and summation is implied over the repeated indices, i and j. The coefficients $T^i_{\circ j}$ are the contravariant–covariant (abbreviated as cnv) components of $\mathbf{T}$.

Projecting expansion (2.5.3) from the left onto $\mathbf{b}^m$, and using the aforementioned orthogonality property, we obtain

$$\mathbf{b}^m \cdot \mathbf{T} = \omega^{(m)}\, T^m_{\circ j}\, \mathbf{b}^j, \tag{2.5.4}$$

where m is a free index. Projecting (2.5.4) onto $\mathbf{b}_n$, using once again the orthogonality property, and rearranging, we obtain an explicit expression for the matrix components,

$$T^m_{\circ n} = \frac{1}{\omega^{(m)}\, \omega^{(n)}}\, \mathbf{b}^m \cdot \mathbf{T} \cdot \mathbf{b}_n, \tag{2.5.5}$$

where n is another free index.

If the two sets of basis vectors are *biorthonormal*, then $\omega^{(m)} = 1$ and $\omega^{(n)} = 1$ for any m and n. Consequently, the fraction on the right-hand side of (2.5.5) is equal to unity.

2.5.3 *Contravariant–covariant base*

We may also consider the contravariant–covariant expansion

$$\mathbf{T} = T_i^{\circ j}\, \mathbf{b}^i \otimes \mathbf{b}_j, \tag{2.5.6}$$

where summation is implied over the repeated indices, i and j, For reasons that will become clear in hindsight, the coefficients $T_i^{\circ j}$ are called the covariant–contravariant (cvn) components of $\mathbf{T}$. Performing projections, we obtain the formula

$$T_m^{\circ n} = \frac{1}{\omega^{(m)}\, \omega^{(n)}}\, \mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}^n, \tag{2.5.7}$$

which differs from that shown in (2.5.5).

2.5.4 Contravariant–contravariant matrix base

As a third possibility, we consider the contravariant–contravariant expansion

$$\mathbf{T} = T_{ij}\,\mathbf{b}^i \otimes \mathbf{b}^j, \tag{2.5.8}$$

where summation is implied over the repeated indices, i and j. The coefficients T_{ij} are the covariant–covariant (cvv) components of $\mathbf{T}$. Performing projections, we obtain

$$T_{mn} = \frac{1}{\omega^{(m)}\,\omega^{(n)}}\,\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n. \tag{2.5.9}$$

Note that, to obtain the cvv components, we employ the covariant base vectors, $\mathbf{b}_m$ and $\mathbf{b}_n$.

2.5.5 Covariant–covariant matrix base

As a fourth possibility, we consider the covariant–covariant expansion

$$\mathbf{T} = T^{ij}\,\mathbf{b}_i \otimes \mathbf{b}_j, \tag{2.5.10}$$

where summation is implied over the repeated indices, i and j. The coefficients T^{ij} are the contravariant–contravariant (cnn) components of $\mathbf{T}$. Performing projections, we obtain

$$T^{mn} = \frac{1}{\omega^{(m)}\,\omega^{(n)}}\,\mathbf{b}^m \cdot \mathbf{T} \cdot \mathbf{b}^n. \tag{2.5.11}$$

Note that, to obtain the cnn components, we employ the contravariant base vectors, $\mathbf{b}^m$ and $\mathbf{b}^n$.

2.5.6 Notational inconsistency

A notational inconsistency has been introduced inadvertently to conform with standard convention. The ijth element of the matrix $\mathbf{T}$ is denoted by T_{ij}, and the ijth covariant–covariant matrix component was also denoted as T_{ij}. For clarity, when necessary, the former may be denoted with Greek indices as $T_{\alpha\beta}$.

2.5.7 *Numerical computation*

The following Matlab code named *bioten*, located in directory Bio of Tunlib, computes four sets of matrix components, concov, covcon, covcov, and concon, using formulas (2.5.5), (2.5.7), (2.5.11) and (2.5.9):

```
%----------
% bcov1 and bcov2 (covariant)
%----------

thbcov1 = 0.034*pi;    % arbitrary
thbcov2 = 0.334*pi;    % arbitrary

lb1 = 1.4;  % arbitrary
lb2 = 1.8;  % arbitrary

bcov1(1) = lb1*cos(thbcov1); bcov1(2) = lb1*sin(thbcov1);
bcov2(1) = lb2*cos(thbcov2); bcov2(2) = lb2*sin(thbcov2);

%---
% bcon1 and bcon2  (contravariant)
%---

thbcon1 = thbcov2 - 0.5*pi;
thbcon2 = thbcov1 + 0.5*pi;

lc1 = 2.4; % arbitrary
lc2 = 1.2; % arbitrary

%---
% projections omega(1) and omega(2)
%---

omg(1) = bcov1*bcon1';
omg(2) = bcov2*bcon2';

%---
% matrix
%---
```

```
T = [ 1 2;   % arbitrary
      3 4];

%---
% con-cov components
%---

Tnv(1,1) = bcon1*T*bcov1'/(omg(1)*omg(1));
Tnv(1,2) = bcon1*T*bcov2'/(omg(1)*omg(2));
Tnv(2,1) = bcon2*T*bcov1'/(omg(2)*omg(1));
Tnv(2,2) = bcon2*T*bcov2'/(omg(2)*omg(2));

%---
% cov-con components
%---

Tvn(1,1) = bcov1*T*bcon1'/(omg(1)*omg(1));
Tvn(1,2) = bcov1*T*bcon2'/(omg(1)*omg(2));
Tvn(2,1) = bcov2*T*bcon1'/(omg(2)*omg(1));
Tvn(2,2) = bcov2*T*bcon2'/(omg(2)*omg(2));

%---
% con-con components
%---

Tnn(1,1) = bcon1*T*bcon1'/(omg(1)*omg(1));
Tnn(1,2) = bcon1*T*bcon2'/(omg(1)*omg(2));
Tnn(2,1) = bcon2*T*bcon1'/(omg(2)*omg(1));
Tnn(2,2) = bcon2*T*bcon2'/(omg(2)*omg(2));

%---
% cov-cov components
%---

Tvv(1,1) = bcov1*T*bcov1'/(omg(1)*omg(1));
Tvv(1,2) = bcov1*T*bcov2'/(omg(1)*omg(2));
Tvv(2,1) = bcov2*T*bcov1'/(omg(2)*omg(1));
Tvv(2,2) = bcov2*T*bcov2'/(omg(2)*omg(2));
```

```
[Tnv Tvn Tnn Tvv]
```

Running the code generates the following output as instructed by the last line of the code:

```
-0.2962   -0.4882   -0.0290    0.7997   -0.3237   -0.1153
                                          0.4149    1.4582
 1.1534    3.3221    0.4212    2.9063    0.3752    1.6202
                                          1.8874    5.7457
```

The first two columns are the components of $\mathbf{T}$ in the covariant-contravariant base, $\mathbf{b}_i \otimes \mathbf{b}^j$. The second pair of columns are the components of $\mathbf{T}$ in the contravariant-covariant base, $\mathbf{b}^i \otimes \mathbf{b}_j$. The third pair of columns are the components of $\mathbf{T}$ in the covariant-covariant base, $\mathbf{b}_i \otimes \mathbf{b}_j$. The fourth pair of columns are the components of $\mathbf{T}$ in the contravariant-contravariant base, $\mathbf{b}^i \otimes \mathbf{b}^j$.

Exercises

2.5.1 Compute the pure and mixed components of the following tensor in a biorthogonal dyadic matrix base with covariant base vectors $\mathbf{b}_1 = [1, 0]$ and $\mathbf{b}_2 = [1, 1]$,

$$\mathbf{T} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}. \tag{2.5.12}$$

2.5.2 Run the code *bioten* for $\mathbf{T}$ equal to the identity matrix, $\mathbf{I}$, and discuss the results.

2.6 Biorthogonal tensor components

We recall the four tensor bases introduced in Section 2.5 in terms of biorthogonal sets of base vectors,

$$\mathbf{b}_i \otimes \mathbf{b}_j, \qquad \mathbf{b}_i \otimes \mathbf{b}^j, \qquad \mathbf{b}^i \otimes \mathbf{b}_j, \qquad \mathbf{b}^i \otimes \mathbf{b}^j. \tag{2.6.1}$$

An arbitrary tensor, $\mathbf{T}$, can be expanded as

$$\mathbf{T} = T^{ij}\,\mathbf{b}_i \otimes \mathbf{b}_j = T^{i}_{\circ j}\,\mathbf{b}_i \otimes \mathbf{b}^j = T_i^{\circ j}\,\mathbf{b}^i \otimes \mathbf{b}_j = T_{ij}\,\mathbf{b}^i \otimes \mathbf{b}^j, \qquad (2.6.2)$$

where summation is implied over the repeated indices i and j.

2.6.1 Four sets of tensor components

The coefficients of $\mathbf{T}$ in the aforementioned bases are named pure contravariant, T^{ij}, contravariant–covariant, $T^{i}_{\circ j}$, covariant–contravariant, $T_i^{\circ j}$, and pure covariant, T_{ij}. We recall that a hollow circle serves as a space holder.

The four sets of coefficients can be arranged into different component matrices. We will see that, if one set is known, the other three sets can be computed by straightforward conversion.

2.6.2 Tensor transpose

The transpose of the matrix $\mathbf{T}$ can be expanded in four ways,

$$\mathbf{T}^{\mathrm{T}} = T^{ji}\,\mathbf{b}_i \otimes \mathbf{b}_j = T_j^{\circ i}\,\mathbf{b}_i \otimes \mathbf{b}^j = T^{j}_{\circ i}\,\mathbf{b}^i \otimes \mathbf{b}_j = T_{ji}\,\mathbf{b}^i \otimes \mathbf{b}^j. \qquad (2.6.3)$$

If the tensor $\mathbf{T}$ is symmetric, then

$$T^{ij} = T^{ji}, \qquad T^{i}_{\circ j} = T_j^{\circ i}, \qquad T_i^{\circ j} = T^{j}_{\circ i}, \qquad T_{ij} = T_{ji}, \qquad (2.6.4)$$

and correspondingly $T_{\alpha\beta} = T_{\beta\alpha}$, where Greek indices correspond to Cartesian coordinates. If the tensor $\mathbf{T}$ is antisymmetric, then

$$T^{ij} = -T^{ji}, \quad T^{i}_{\circ j} = -T_j^{\circ i}, \quad T_i^{\circ j} = -T^{j}_{\circ i}, \quad T_{ij} = -T_{ji} \qquad (2.6.5)$$

and correspondingly $T_{\alpha\beta} = -T_{\beta\alpha}$.

2.6.3 Conversion

Projecting equations (2.6.2) from the right on $\mathbf{b}^n$, where n is a free index, we obtain the vector equation

$$\mathbf{T}\cdot\mathbf{b}^n = T^{in}\omega^{(n)}\,\mathbf{b}_i = T^{i}_{\circ j}\,b^{jn}\mathbf{b}_i = T_i^{\circ n}\omega^{(n)}\,\mathbf{b}^i = T_{ij}\,b^{jn}\,\mathbf{b}^i. \qquad (2.6.6)$$

Projecting equation (2.6.6) onto $\mathbf{b}^m$, where m is a free index, we obtain the scalar equation

$$\begin{aligned}\mathbf{b}^m \cdot \mathbf{T} \cdot \mathbf{b}^n &= T^{mn} \omega^{(n)} \, \omega^{(m)} = T^m_{\circ j} \, \omega^{(m)} \, b^{jn} \\ &= T_i^{\circ n} \omega^{(n)} \, b^{im} = T_{ij} \, b^{im} \, b^{jn}. \end{aligned} \tag{2.6.7}$$

Rearranging, we obtain

$$\begin{aligned}T^{mn} &= \frac{1}{\omega^{(n)}} \, T^m_{\circ j} \, b^{jn} = \frac{1}{\omega^{(m)}} \, T_i^{\circ n} \, b^{im} \\ &= \frac{1}{\omega^{(n)} \, \omega^{(m)}} \, T_{ij} \, b^{im} \, b^{jn}, \end{aligned} \tag{2.6.8}$$

which provides us with a formula for raising one or two indices.

2.6.4 Conversion continued

We continue the conversion process by projecting equation (2.6.6) on $\mathbf{b}_m$, where m is a free index, to obtain

$$\begin{aligned}\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}^n &= T^{in} \omega^{(n)} \, b_{im} = T^i_{\circ j} \, b^{jn} \, b_{im} \\ &= T_m^{\circ n} \omega^{(n)} \omega^{(m)} = T_{mj} \, \omega^{(m)} \, b^{jn}. \end{aligned} \tag{2.6.9}$$

Rearranging, we obtain

$$T_m^{\circ n} = \frac{1}{\omega^{(m)}} \, T^{in} \, b_{im} = \frac{1}{\omega^{(n)}} \, T_{mj} \, b^{jn} = \frac{1}{\omega^{(n)} \omega^{(m)}} \, T^i_{\circ j} \, b_{im} \, b^{jn}, \tag{2.6.10}$$

which provides us with a formula for raising one or two indices.

2.6.5 Confirmation by code

The following lines of code, continuing code *bioten* listed in Section 2.3, computes the cvn components, $T_m^{\circ n}$, from the cnv components, $T^m_{\circ n}$, using the formula

$$T_m^{\circ n} = \frac{1}{\omega^{(n)} \omega^{(m)}} \, T^i_{\circ j} \, b_{im} \, b^{jn}, \tag{2.6.11}$$

where summation is implied over the repeated indices, i and j:

```
%---
% compute covmet (b)
%---

covmet(1,1) = bcov1*bcov1'; covmet(1,2) = bcov1*bcov2';
covmet(2,1) = bcov2*bcov1'; covmet(2,2) = bcov2*bcov2';

%---
% compute conmet (beta)
%---

conmet(1,1) = bcon1*bcon1'; conmet(1,2) = bcon1*bcon2';
conmet(2,1) = bcon2*bcon1'; conmet(2,2) = bcon2*bcon2';

%---
% lower and raise an index
%
% vn: covariant-contravariant
% nv: contravariant-covariant
%---

for m=1:2
 for n=1:2
  Tvn1(m,n) = 0.0;
  for i=1:2
   for j=1:2
    Tvn1(m,n) = Tvn1(m,n) + Tnv(i,j)*covmet(i,m)*conmet(j,n);
   end
  end
  Tvn1(m,n) = Tvn1(m,n)/(omg(m)*omg(n));
 end
end

[Tvn Tvn1]
```

Running the code generates the following output, as instructed by the last line of the code:

```
   -0.0290    0.7997   -0.0290    0.7997
    0.4212    2.9063    0.4212    2.9063
```

thereby confirming the conversion formulas.

2.6.6 Determinants

The following relations can be established between the determinants,

$$\det(\mathbf{T}) = \det(\boldsymbol{\omega}) \times \det[T^i_{\circ j}] = \det[\boldsymbol{\omega}] \times \det[T_i^{\circ j}] \qquad (2.6.12)$$

and also

$$\det(\mathbf{T}) = \det(\mathbf{b}) \times \det[T^{ij}] = \det(\boldsymbol{\beta}) \times \det[T_{ij}], \qquad (2.6.13)$$

where

$$\det(\boldsymbol{\omega}) = \prod_{i=1}^{N} \omega^{(i)}. \qquad (2.6.14)$$

Only one of these formulas is necessary to be established, and the others arise from the transformation rules for the contravariant and covariant tensor components.

2.6.7 Confirmation by code

The following Matlab code, continuing the code *bioten* listed previously in this section, confirms these formulas for the determinant:

```
%---
% four matrix bases
%---

for i=1:2
 for j=1:2

  Bconcov11(i,j) = bcon1(i)*bcov1(j);
  Bconcov12(i,j) = bcon1(i)*bcov2(j);
  Bconcov21(i,j) = bcon2(i)*bcov1(j);
  Bconcov22(i,j) = bcon2(i)*bcov2(j);

  Bcovcon11(i,j) = bcov1(i)*bcon1(j);
  Bcovcon12(i,j) = bcov1(i)*bcon2(j);
  Bcovcon21(i,j) = bcov2(i)*bcon1(j);
```

```
  Bcovcon22(i,j) = bcov2(i)*bcon2(j);

  Bcovcov11(i,j) = bcov1(i)*bcov1(j);
  Bcovcov12(i,j) = bcov1(i)*bcov2(j);
  Bcovcov21(i,j) = bcov2(i)*bcov1(j);
  Bcovcov22(i,j) = bcov2(i)*bcov2(j);

  Bconcon11(i,j) = bcon1(i)*bcon1(j);
  Bconcon12(i,j) = bcon1(i)*bcon2(j);
  Bconcon21(i,j) = bcon2(i)*bcon1(j);
  Bconcon22(i,j) = bcon2(i)*bcon2(j);

 end
end

%---
% determinants
%---

A(1,1) = 0.4; A(1,2) = 0.1;  % arbitrary matrix
A(2,1) = 0.8; A(2,2) = 0.7;

detA = det(A)

% some matrix

AVV = A(1,1)*Bcovcov11+A(1,2)*Bcovcov12 ...
     +A(2,1)*Bcovcov21+A(2,2)*Bcovcov22;

% another matrix

ANN = A(1,1)*Bconcon11+A(1,2)*Bconcon12 ...
     +A(2,1)*Bconcon21+A(2,2)*Bconcon22;

% another matrix

ANV = A(1,1)*Bconcov11+A(1,2)*Bconcov12 ...
     +A(2,1)*Bconcov21+A(2,2)*Bconcov22;
```

```
% another matrix

AVN = A(1,1)*Bcovcon11+A(1,2)*Bcovcon12 ...
     +A(2,1)*Bcovcon21+A(2,2)*Bcovcon22;

% determinants in terms of det(A)

detomg = omg(1)*omg(2);
detcov = covmet(1,1)*covmet(2,2)-covmet(1,2)^2;
detcon = conmet(1,1)*conmet(2,2)-conmet(1,2)^2;

[det(AVV) detA*detcov;
 det(ANN) detA*detcon;
 det(ANV) detA*detomg;
 det(AVN) detA*detomg]
```

Running the code generates the following output:

```
    0.8313    0.8313
    1.0858    1.0858
    0.9500    0.9500
    0.9500    0.9500
```

as prompted by the last four lines of the code. We see that, as expected, the two columns are identical.

Exercises

2.6.1 Explain why the determinant of diagonal matrix is the product of the diagonal elements.

2.6.2 Derive expressions for the trace of a tensor in terms of its orthogonal base components.

2.7 Tensor multiplication

Tensors can multiply vectors and other tensors in several ways. The results can be expressed in compact form in terms of contravariant and covariant components.

2.7.1 *Product of a tensor with a vector,* Tv

The product of a tensor, $\mathbf{T}$, with a vector $\mathbf{v}$, is a new vector given by

$$\mathbf{u} \equiv \mathbf{T} \cdot \mathbf{v} = (T^{i}_{\circ j}\, \mathbf{b}_i \otimes \mathbf{b}^j) \cdot (v^m\, \mathbf{b}_m). \tag{2.7.1}$$

We find that

$$\mathbf{u} \equiv \mathbf{T} \cdot \mathbf{v} = T^{i}_{\circ j}\, v^m (\mathbf{b}_i \otimes \mathbf{b}^j) \cdot \mathbf{b}_m = T^{i}_{\circ j}\, v^m\, \mathbf{b}_i\, (\mathbf{b}^j \cdot \mathbf{b}_m), \tag{2.7.2}$$

yielding

$$\mathbf{u} \equiv \mathbf{T} \cdot \mathbf{v} = T^{i}_{\circ j}\, v^j\, \omega^{(j)}\, \mathbf{b}_i, \tag{2.7.3}$$

which shows that

$$u^i = T^{i}_{\circ j}\, v^j\, \omega^{(j)}. \tag{2.7.4}$$

Working in a similar fashion, we find that

$$u^i = T^{ij}\, v_j\, \omega^{(j)} \tag{2.7.5}$$

and

$$u_i = T_{ij}\, v^j\, \omega^{(j)} = T_{i}^{\circ j}\, v_j\, \omega^{(j)} \tag{2.7.6}$$

for the covariant vector components.

2.7.2 *Product of a tensor with a vector,* vT

Repeating the preceding procedure for vector–matrix multiplication, now we define the vector $\mathbf{h} = \mathbf{v} \cdot \mathbf{T}$, and find that

$$h^i = T^{ji}\, v_j\, \omega^{(j)} = T_{j}^{\circ i}\, v^j\, \omega^{(j)} \tag{2.7.7}$$

for the contravariant components and

$$h_i = T_{ji}\, v^j\, \omega^{(j)} = T^{j}_{\circ i}\, v_j\, \omega^{(j)} \tag{2.7.8}$$

for the covariant components.

2.7.3 Product of two tensors

The product of two tensors, $\mathbf{T}$ and $\mathbf{S}$, is another tensor given by

$$\mathbf{W} \equiv \mathbf{T} \cdot \mathbf{S} = (T^i_{\circ j}\, \mathbf{b}_i \otimes \mathbf{b}^j) \cdot (S^{mn}\, \mathbf{b}_m \otimes \mathbf{b}_n). \tag{2.7.9}$$

We find that

$$\begin{aligned} \mathbf{W} &= T^i_{\circ j}\, S^{mn}\, (\mathbf{b}_i \otimes \mathbf{b}^j) \cdot (\mathbf{b}_m \otimes \mathbf{b}_n) \\ &= T^i_{\circ j}\, S^{mn}\, (\mathbf{b}^j \cdot \mathbf{b}_m)\, \mathbf{b}_i \otimes \mathbf{b}_n, \end{aligned} \tag{2.7.10}$$

yielding

$$\mathbf{W} = T^i_{\circ j} S^{jn}\, \omega^{(j)}\, \mathbf{b}_i \otimes \mathbf{b}_n. \tag{2.7.11}$$

The pure contravariant (cnn) components of $\mathbf{W}$ are thus given by

$$W^{in} = T^i_{\circ j}\, S^{jn}\, \omega^{(j)}, \tag{2.7.12}$$

where summation is implied over the repeated index, j. Working in a similar fashion, we find that

$$W_{in} = T_i^{\circ j}\, S^j_{\circ n}\, \omega^{(j)}, \qquad W_i^{\circ n} = T_{ij}\, S^{jn}\, \omega^{(j)}, \qquad W^i_{\circ n} = T^{ij}\, S_{jn}\, \omega^{(j)}, \tag{2.7.13}$$

involving the cvn, cnv, cvv, and cnn components of $\mathbf{T}$ and $\mathbf{S}$, where summation is implied over the repeated index, j.

2.7.4 Double-dot product

The double-dot product of two tensors, $\mathbf{T}$ and $\mathbf{S}$, is a scalar given by

$$\mathbf{T} : \mathbf{S} \equiv \mathrm{trace}(\mathbf{T}^{\mathrm{T}} \cdot \mathbf{S}) = \mathrm{trace}(\mathbf{T} \cdot \mathbf{S}^{\mathrm{T}}), \tag{2.7.14}$$

where the superscript T denotes the matrix transpose. Using the last formula in (2.7.13), we find that

$$\mathbf{T}^{\mathrm{T}} \cdot \mathbf{S} = T^{ji}\, S_{jn}\, \omega^{(j)}\, \mathbf{b}_i \otimes \mathbf{b}^n. \tag{2.7.15}$$

Next, we note that

$$\mathrm{trace}(\mathbf{b}_i \otimes \mathbf{b}^n) = \omega^{(i)}\, \delta_{in}, \tag{2.7.16}$$

and obtain

$$\mathbf{T} : \mathbf{S} = T_{ij} S^{ij} \, \omega^{(i)} \, \omega^{(j)} = T^{ij} S_{ij} \, \omega^{(i)} \, \omega^{(j)}, \tag{2.7.17}$$

where summation is implied over the repeated indices, i and j.

Exercise

2.7.1 Derive expression (2.7.17).

2.8 Resolution of the identity tensor

Equation (2.5.5) provides us with an expression for the contravariant–covariant (cnv) components of an arbitrary tensor, $\mathbf{T}$,

$$T^m_{\text{on}} = \frac{1}{\omega^{(m)} \, \omega^{(n)}} \, \mathbf{b}^m \cdot \mathbf{T} \cdot \mathbf{b}_n. \tag{2.8.1}$$

Identifying $\mathbf{T}$ with the identity tensor, $\mathbf{I}$, we find that

$$I^m_{\text{on}} = \frac{1}{\omega^{(m)}} \, \delta_{mn} \tag{2.8.2}$$

and obtain

$$\mathbf{I} = \frac{1}{\omega^{(i)}} \, \mathbf{b}_i \otimes \mathbf{b}^i, \tag{2.8.3}$$

where summation is implied over the repeated index i.

Working in a similar fashion, we obtain the expansion

$$\mathbf{I} = \frac{1}{\omega^{(i)}} \, \mathbf{b}^i \otimes \mathbf{b}_i, \tag{2.8.4}$$

where summation is implied over the repeated index i.

Substituting into (2.8.3) the expression

$$\mathbf{b}^i = \frac{1}{\omega^{(j)}} \, b^{ij} \, \mathbf{b}_j, \tag{2.8.5}$$

we find that

$$\mathbf{I} = \frac{1}{\omega^{(i)}\omega^{(j)}}\, b^{ij}\, \mathbf{b}_i \otimes \mathbf{b}_j. \qquad (2.8.6)$$

Substituting into (2.8.4) the expression

$$\mathbf{b}_i = \frac{1}{\omega^{(j)}}\, b_{ij}\, \mathbf{b}^j, \qquad (2.8.7)$$

we find that

$$\mathbf{I} = \frac{1}{\omega^{(i)}\omega^{(j)}}\, b_{ij}\, \mathbf{b}^i \otimes \mathbf{b}^j, \qquad (2.8.8)$$

where summation is implied over the repeated indices i and j.

Compiling equations (2.8.3), (2.8.4), (2.8.6), and (2.8.8), we obtain a four-fold expansion,

$$\mathbf{I} = \frac{1}{\omega^{(i)}\omega^{(j)}}\, b^{ij}\, \mathbf{b}_i \otimes \mathbf{b}_j = \frac{1}{\omega^{(i)}}\, \mathbf{b}_i \otimes \mathbf{b}^i = \frac{1}{\omega^{(i)}}\, \mathbf{b}^i \otimes \mathbf{b}_i = \frac{1}{\omega^{(i)}\omega^{(j)}}\, b_{ij}\, \mathbf{b}^i \otimes \mathbf{b}^j, \qquad (2.8.9)$$

where summation is implied over the repeated indices, i and j.

2.8.1 Confirmation by code

The following Matlab code named *bioid*, located in directory Bio of Tunlib, confirms this four-fold identity:

```
%----------
% bcov1 and bcov2 (covariant)
%----------

thbcov1 = 0.034*pi;    % arbitrary
thbcov2 = 0.334*pi;    % arbitrary

lb1 = 1.4;  % arbitrary
lb2 = 1.8;  % arbitrary

bcov1(1) = lb1*cos(thbcov1); bcov1(2) = lb1*sin(thbcov1);
```

```
bcov2(1) = lb2*cos(thbcov2); bcov2(2) = lb2*sin(thbcov2);

%---
% bcon1 and bcon2  (contravariant)
%---

thbcon1 = thbcov2 - 0.5*pi;
thbcon2 = thbcov1 + 0.5*pi;

lc1 = 2.4; % arbitrary
lc2 = 1.2; % arbitrary

bcon1(1) = lc1*cos(thbcon1); bcon1(2) = lc1*sin(thbcon1);
bcon2(1) = lc2*cos(thbcon2); bcon2(2) = lc2*sin(thbcon2);

%---
% projections
%---

omg(1) = bcov1*bcon1';
omg(2) = bcov2*bcon2';

%---
% compute covmet
% (covariant metric coefficients)
%---

covmet(1,1) = bcov1*bcov1'; covmet(1,2) = bcov1*bcov2';
covmet(2,1) = bcov2*bcov1'; covmet(2,2) = bcov2*bcov2';

%---
% compute conmet
% (contravariant metric coefficients)
%---

conmet(1,1) = bcon1*bcon1'; conmet(1,2) = bcon1*bcon2';
conmet(2,1) = bcon2*bcon1'; conmet(2,2) = bcon2*bcon2';

%---
```

```
% concon and covcov components of the identity matrix
%---

for i=1:2
 for j=1:2
   Iconcon(i,j) = conmet(i,j)/(omg(i)*omg(j));
   Icovcov(i,j) = covmet(i,j)/(omg(i)*omg(j));
 end
end

%---
% four matrix bases
%---

for i=1:2
 for j=1:2

  Bconcov11(i,j) = bcon1(i)*bcov1(j);
  Bconcov12(i,j) = bcon1(i)*bcov2(j);
  Bconcov21(i,j) = bcon2(i)*bcov1(j);
  Bconcov22(i,j) = bcon2(i)*bcov2(j);

  Bcovcon11(i,j) = bcov1(i)*bcon1(j);
  Bcovcon12(i,j) = bcov1(i)*bcon2(j);
  Bcovcon21(i,j) = bcov2(i)*bcon1(j);
  Bcovcon22(i,j) = bcov2(i)*bcon2(j);

  Bcovcov11(i,j) = bcov1(i)*bcov1(j);
  Bcovcov12(i,j) = bcov1(i)*bcov2(j);
  Bcovcov21(i,j) = bcov2(i)*bcov1(j);
  Bcovcov22(i,j) = bcov2(i)*bcov2(j);

  Bconcon11(i,j) = bcon1(i)*bcon1(j);
  Bconcon12(i,j) = bcon1(i)*bcon2(j);
  Bconcon21(i,j) = bcon2(i)*bcon1(j);
  Bconcon22(i,j) = bcon2(i)*bcon2(j);

 end
end
```

```
%---
% identities
%---

Identity1 =  Iconcon(1,1)*Bcovcov11 ...
           + Iconcon(1,2)*Bcovcov12 ...
           + Iconcon(2,1)*Bcovcov21 ...
           + Iconcon(2,2)*Bcovcov22;
Identity2 =  Icovcov(1,1)*Bconcon11 ...
           + Icovcov(1,2)*Bconcon12 ...
           + Icovcov(2,1)*Bconcon21 ...
           + Icovcov(2,2)*Bconcon22;
Identity3 =  1.0/omg(1) *Bconcov11 ...
            + 1.0/omg(2)*Bconcov22;
Identity4 =  1.0/omg(1) *Bcovcon11 ...
            + 1.0/omg(2)*Bcovcon22;

[Identity1 Identity2 Identity3 Identity4]
```

Running the code generates the following output prompted by the last line of the code:

```
1.0000    0.0000    1.0000    0.0000    1.0000    0.0000
                                        1.0000    0.0000
0.0000    1.0000    0.0000    1.0000    0.0000    1.0000
                                        0.0000    1.0000
```

The output consists of four identity matrices printed alongside.

2.9 *Tensor inverse*

The inverse of a tensor, $\mathbf{T}$, denoted by

$$\mathbf{S} = \mathbf{T}^{-1}, \tag{2.9.1}$$

satisfies (2.7.9) with $\mathbf{W} = \mathbf{I}$,

$$\mathbf{I} \equiv \mathbf{T} \cdot \mathbf{S}, \tag{2.9.2}$$

where $\mathbf{I}$ is the identity tensor. Recalling the representation of the identity tensor shown in (2.8.9), we find that

$$T_{in} S^{nj} \, \omega^{(n)} = \frac{1}{\omega^{(i)}} \delta_{ij}, \qquad T^{in} S_{nj} \, \omega^{(n)} = \frac{1}{\omega^{(i)}} \delta_{ij}, \qquad (2.9.3)$$

and also

$$T_i^{\circ n} S_{nj} \, \omega^{(n)} = \frac{1}{\omega^{(i)} \omega^{(j)}} \, b_{ij}, \qquad T^i_{\circ n} S^{nj} \, \beta^{(n)} = \frac{1}{\omega^{(i)} \omega^{(j)}} \, b^{ij}, \quad (2.9.4)$$

where summation is implied over the repeated index, n.

2.9.1 Confirmation by code

The following lines of code, continuing code *bioten* listed in Sections 2.3 and 2.4 confirms these formulas:

```
S = inv(T);

%---
% con-cov components
%---

Snv(1,1) = bcon1*S*bcov1'/(omg(1)*omg(1));
Snv(1,2) = bcon1*S*bcov2'/(omg(1)*omg(2));
Snv(2,1) = bcon2*S*bcov1'/(omg(2)*omg(1));
Snv(2,2) = bcon2*S*bcov2'/(omg(2)*omg(2));

%---
% cov-con components
%---

Svn(1,1) = bcov1*S*bcon1'/(omg(1)*omg(1));
Svn(1,2) = bcov1*S*bcon2'/(omg(1)*omg(2));
Svn(2,1) = bcov2*S*bcon1'/(omg(2)*omg(1));
Svn(2,2) = bcov2*S*bcon2'/(omg(2)*omg(2));

%---
% con-con components
%---
```

```
Snn(1,1) = bcon1*S*bcon1'/(omg(1)*omg(1));
Snn(1,2) = bcon1*S*bcon2'/(omg(1)*beta(2));
Snn(2,1) = bcon2*S*bcon1'/(omg(2)*omg(1));
Snn(2,2) = bcon2*S*bcon2'/(omg(2)*omg(2));

%---
% cov-cov components
%---

Svv(1,1) = bcov1*S*bcov1'/(omg(1)*omg(1));
Svv(1,2) = bcov1*S*bcov2'/(omg(1)*omg(2));
Svv(2,1) = bcov2*S*bcov1'/(omg(2)*omg(1));
Svv(2,2) = bcov2*S*bcov2'/(omg(2)*omg(2));

%---
% test and confirm
%---

for i=1:2
 for j=1:2
  test1(i,j) = 0.0;
  test2(i,j) = 0.0;
  test3(i,j) = 0.0;
  test4(i,j) = 0.0;
  for n=1:2
   test1(i,j) = test1(i,j)+Tvn(i,n)*Svv(n,j)*beta(n);
   test2(i,j) = test2(i,j)+Tnv(i,n)*Snn(n,j)*beta(n);
   test3(i,j) = test3(i,j)+Tnn(i,n)*Svv(n,j)*beta(n);
   test4(i,j) = test4(i,j)+Tvv(i,n)*Snn(n,j)*beta(n);
  end
  verify1(i,j) = covmet(i,j)/beta(i)/beta(j);
  verify2(i,j) = conmet(i,j)/beta(i)/beta(j);
 end
  verify3(i,i) = 1.0/beta(i);
  verify4(i,i) = 1.0/beta(i);
end

%---
% test should be equal to confirm
```

```
%---

[test1 verify1;
 test2 verify2;
 test3 verify3;
 test4 verify4
]
```

Running the code generates the following output prompted by the last line:

```
    0.2653    0.3118    0.2653    0.3118
    0.3118    1.0610    0.3118    1.0610

    0.7795   -0.3564    0.7795   -0.3564
   -0.3564    0.4716   -0.3564    0.4716

    0.3679    0.0000    0.3679         0
    0.0000    0.5723         0    0.5723

    0.3679         0    0.3679         0
         0    0.5723         0    0.5723
```

The 2×2 matrices on the left are equal to those on the right, as required.

Exercise

2.9.1 Run the code given in the text for different base vectors of your choice and confirm the accuracy of the results.

2.10 Diagonal component matrix

Suppose that the covariant arrays, $\mathbf{b}_1, \ldots, \mathbf{b}_N$, are eigenvectors of a square $N \times N$ matrix, $\mathbf{T}$, with eigenvalues $\lambda^{(n)}$, so that

$$\mathbf{T} \cdot \mathbf{b}_n = \lambda^{(n)} \, \mathbf{b}_n. \tag{2.10.1}$$

The associated contravariant arrays, $\mathbf{b}^1, \ldots, \mathbf{b}^N$, are the left eigenvectors of $\mathbf{T}$, satisfying the orthogonality condition

$$\mathbf{b}^m \cdot \mathbf{b}_n = 0. \tag{2.10.2}$$

Generalized eigenvectors can be introduced in the case of multiple eigenvalues, as necessary.

2.10.1 *cnv components*

Consider the covariant–contravariant expansion

$$\mathbf{T} = T^m_{\text{on}}\, \mathbf{b}_m \otimes \mathbf{b}^n, \tag{2.10.3}$$

where summation is implied over the repeated indices m and n. Equation (2.5.5), repeated below for convenience,

$$T^m_{\text{on}} = \frac{1}{\omega^{(m)}\omega^{(n)}}\, \mathbf{b}^m \cdot \mathbf{T} \cdot \mathbf{b}_n, \tag{2.10.4}$$

yields

$$T^m_{\text{on}} = \frac{1}{\omega^{(m)}\omega^{(n)}}\, \lambda^{(n)}\, \mathbf{b}^m \cdot \mathbf{b}_n, \tag{2.10.5}$$

where summation is *not* implied over n or m. Simplifying, we obtain the coefficients

$$T^m_{\text{on}} = \frac{1}{\omega^{(m)}}\, \lambda^{(m)}\, \delta_{nm}, \tag{2.10.6}$$

where δ_{nm} is Kronecker's delta. We conclude that the contravariant-covariant (cnv) component matrix is a diagonal matrix with diagonal elements given by

$$T^m_{\text{om}} = \frac{1}{\omega^{(m)}}\, \lambda^{(m)}. \tag{2.10.7}$$

When $\omega^{(m)} = 1$ for any m, the matrix T^m_{on} is the diagonal matrix of eigenvalues.

The expansion (2.10.3) takes the spectral form

$$\mathbf{T} = \frac{1}{\omega^{(m)}}\, \lambda^{(m)}\, \mathbf{b}_m \otimes \mathbf{b}^m, \tag{2.10.8}$$

where summation is implied over the repeated index, m. To confirm this expansion, we compute

$$\mathbf{T}\cdot\mathbf{b}_n = \frac{1}{\omega^{(m)}}\lambda^{(m)}\,\mathbf{b}_m\,(\mathbf{b}^m\cdot\mathbf{b}_n) = \lambda^{(n)}\,\mathbf{b}_n, \tag{2.10.9}$$

where n is a free index, which shows that $\mathbf{b}_n$ is an eigenvector of $\mathbf{T}$ with corresponding eigenvalue $\lambda^{(n)}$.

2.10.2 cvn components

Using the rules for raising and lowering indices, we find that

$$T_m^{\circ n} = \frac{1}{\omega^{(n)}\omega^{(m)}}\,T^i_{\circ j}\,b_{im}\,b^{jn} = \frac{1}{\omega^{(n)}\omega^{(m)}}\,\frac{1}{\omega^{(i)}}\,\lambda^{(i)}\,b_{im}\,b^{in}, \tag{2.10.10}$$

where summation is implied over the repeated index, i.

2.10.3 cnn components

Using the rules for raising an index, we find that

$$T^{mn} = \frac{1}{\omega^{(n)}}\,T^m_{\circ j}\,b^{jn} = \frac{1}{\omega^{(n)}\omega^{(m)}}\,\lambda^{(m)}\,b^{mn}. \tag{2.10.11}$$

2.10.4 cvv components

Using the rules for lowering an index, we find that

$$T_{mn} = \frac{1}{\omega^{(n)}}\,T^j_{\circ n}\,b_{jm} = \frac{1}{\omega^{(n)}\omega^{(m)}}\,\lambda^{(n)}\,b_{mn}. \tag{2.10.12}$$

2.10.5 Confirmation by code

The following lines of code named *biodiag*, located in directory Bio of Tunlib, confirms that formulas derived in this section:

```
%---
% specify a matrix
%---

T = [ 1.1, 2.1;
```

```
      0.4, 3.6];

%---
% compute eigenvectors and eigenvalues
%---

[EIG,  LAM] = eig(T);
[EIGT, LAM] = eig(T');

lam(1) = LAM(1,1);
lam(2) = LAM(2,2);

%----------
% bcov1 and bcov2 (covariant)
%----------

fc1 = 2.3; % arbitrary
fc2 = 1.4; % arbitrary

bcov1(1) = fc1*EIG(1,1);
bcov1(2) = fc1*EIG(2,1);
bcov2(1) = fc2*EIG(1,2);
bcov2(2) = fc2*EIG(2,2);

%---
% bcon1 and bcon2  (contravariant)
%---

fc1 =-0.7; % arbitrary
fc2 = 2.4; % arbitrary

bcon1(1) = fc1*EIGT(1,1);
bcon1(2) = fc1*EIGT(2,1);
bcon2(1) = fc2*EIGT(1,2);
bcon2(2) = fc2*EIGT(2,2);

%---
% projections omega(1) and omega(2)
%---
```

```
omg(1) = bcov1*bcon1';
omg(2) = bcov2*bcon2';

%---
% con-cov components
%---

Tnv(1,1) = bcon1*T*bcov1'/(omg(1)*omg(1));
Tnv(1,2) = bcon1*T*bcov2'/(omg(1)*omg(2));
Tnv(2,1) = bcon2*T*bcov1'/(omg(2)*omg(1));
Tnv(2,2) = bcon2*T*bcov2'/(omg(2)*omg(2));

%---
% cov-con components
%---

Tvn(1,1) = bcov1*T*bcon1'/(omg(1)*omg(1));
Tvn(1,2) = bcov1*T*bcon2'/(omg(1)*omg(2));
Tvn(2,1) = bcov2*T*bcon1'/(omg(2)*omg(1));
Tvn(2,2) = bcov2*T*bcon2'/(omg(2)*omg(2));

%---
% con-con components
%---

Tnn(1,1) = bcon1*T*bcon1'/(omg(1)*omg(1));
Tnn(1,2) = bcon1*T*bcon2'/(omg(1)*omg(2));
Tnn(2,1) = bcon2*T*bcon1'/(omg(2)*omg(1));
Tnn(2,2) = bcon2*T*bcon2'/(omg(2)*omg(2));

%---
% cov-cov components
%---

Tvv(1,1) = bcov1*T*bcov1'/(omg(1)*omg(1));
Tvv(1,2) = bcov1*T*bcov2'/(omg(1)*omg(2));
Tvv(2,1) = bcov2*T*bcov1'/(omg(2)*omg(1));
Tvv(2,2) = bcov2*T*bcov2'/(omg(2)*omg(2));
```

```
[Tnn, Tvv, Tnv, Tvn]

%---
% compute covmet (b)
%---

covmet(1,1) = bcov1*bcov1';
covmet(1,2) = bcov1*bcov2';
covmet(2,1) = bcov2*bcov1';
covmet(2,2) = bcov2*bcov2';

%---
% compute conmet (beta)
%---

conmet(1,1) = bcon1*bcon1';
conmet(1,2) = bcon1*bcon2';
conmet(2,1) = bcon2*bcon1';
conmet(2,2) = bcon2*bcon2';

%---
% in terms of eigenvalues
%---

for m=1:2
 for n=1:2
  den = omg(m)*omg(n);
  Tnv1(m,n) = 0.0;
  Tnn1(m,n) = lam(m)*conmet(m,n)/den;
  Tvv1(m,n) = lam(n)*covmet(m,n)/den;
  Tvn1(m,n) = covmet(1,m)*conmet(1,n)*lam(1)/omg(1) ...
            + covmet(2,m)*conmet(2,n)*lam(2)/omg(2);
  Tvn1(m,n) = Tvn1(m,n)/den;
 end
  Tnv1(m,m) = lam(m)/omg(m);
end

[Tnv Tnv1;
```

```
 Tnn Tnn1;
 Tvn Tvn1;
 Tvv Tvv1]
```

Running the code generates the following output:

```
   -0.5667   -0.0000   -0.5667         0
   -0.0000    1.3238         0    1.3238

    0.1967   -0.1554    0.1967   -0.1554
   -0.7575    2.5882   -0.7575    2.5882

    0.0937   -2.2564    0.0937   -2.2564
   -0.4006    1.6402   -0.4006    1.6402

    2.1236   -1.4519    2.1236   -1.4519
   -0.2978    0.8807   -0.2978    0.8807
```

The first couple of columns is equal to the second couple of columns, as required.

Exercise

2.10.1 Prove that the eigenvectors and left eigenvectors of a matrix corresponding to different eigenvalues are orthogonal.

2.11 Base transformations

Consider a covariant base, $\mathbf{b}_i$, and the associated contravariant base, $\mathbf{b}^i$, and another covariant base, $\widetilde{\mathbf{b}}_i$, and the associated contravariant base, $\widetilde{\mathbf{b}}^i$, where the second base and its properties are indicated by a tilde.

2.11.1 Transformation of covariant base vectors

The two covariant bases are related by the linear equations

$$\widetilde{\mathbf{b}}_i = H_{ij}\,\mathbf{b}_j, \qquad \mathbf{b}_i = H^{-1}_{ij}\,\widetilde{\mathbf{b}}_j, \tag{2.11.1}$$

where $\mathbf{H}$ is a transformation matrix. The matrix $\mathbf{H}^{-1}$ with elements H_{ij}^{-1} is the inverse of the matrix $\mathbf{H}$ with elements H_{ij}.

Projecting equations (2.11.1) onto contravariant untilded and tilded base vectors, we find that

$$H_{ij} = \frac{1}{\omega^{(j)}}\,\widetilde{\mathbf{b}}_i \cdot \mathbf{b}^j, \qquad H_{ij}^{-1} = \frac{1}{\widetilde{\omega}^{(j)}}\,\mathbf{b}_i \cdot \widetilde{\mathbf{b}}^j. \tag{2.11.2}$$

Note that the elements of the matrix $\mathbf{H}$ and its inverse remain constant under a change of the frame of reference.

2.11.2 Matrix formulation

The covariant base vectors $\mathbf{b}_i$ can be arranged at the *columns* of a matrix, $\mathbf{F}$, and the covariant vectors $\widetilde{\mathbf{b}}_i$ can be are arranged at the *columns* of another matrix, $\widetilde{\mathbf{F}}$,

$$\mathbf{F} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \qquad \widetilde{\mathbf{F}} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \widetilde{\mathbf{b}}_1 & \cdots & \widetilde{\mathbf{b}}_N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}. \tag{2.11.3}$$

By definition,

$$\widetilde{\mathbf{F}} = \mathbf{F} \cdot \mathbf{H}^{\mathrm{T}}, \qquad \mathbf{F} = \widetilde{\mathbf{F}} \cdot \mathbf{H}^{-\mathrm{T}}, \qquad \mathbf{H} = \widetilde{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{F}^{-\mathrm{T}}, \tag{2.11.4}$$

where the matrix $\mathbf{H}$ is defined in (2.11.1) and the superscript $-\mathrm{T}$ denotes the inverse of the transpose, which is equal to the transpose of the inverse.

We recall the representations $\mathbf{F} = \mathbf{b}_k \otimes \boldsymbol{\epsilon}_k$ and $\widetilde{\mathbf{F}} = \widetilde{\mathbf{b}}_i \otimes \boldsymbol{\epsilon}_i$, and confirm that

$$\begin{aligned} \mathbf{F} \cdot \mathbf{H}^{\mathrm{T}} &= (\mathbf{b}_k \otimes \boldsymbol{\epsilon}_k) \cdot (H_{ij} \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_i) \\ &= H_{ij} \mathbf{b}_j \otimes \boldsymbol{\epsilon}_i = \widetilde{\mathbf{b}}_i \otimes \boldsymbol{\epsilon}_i = \widetilde{\mathbf{F}}, \end{aligned} \tag{2.11.5}$$

as shown in (2.11.4).

2.11.3 Transformation of contravariant base vectors

The two contravariant bases are related by

$$\widetilde{\mathbf{b}}^i = R_{ij} \mathbf{b}^j, \qquad \mathbf{b}^i = R_{ij}^{-1} \widetilde{\mathbf{b}}^j, \tag{2.11.6}$$

where $\mathbf{R}$ is another base transformation matrix and $\mathbf{R}^{-1}$ is its inverse with elements

$$R_{ij} = \frac{1}{\omega^{(j)}}\, \widetilde{\mathbf{b}}^i \cdot \mathbf{b}_j, \qquad R^{-1}_{ij} = \frac{1}{\widetilde{\omega}^{(j)}}\, \mathbf{b}^i \cdot \widetilde{\mathbf{b}}_j. \tag{2.11.7}$$

These expressions arise by projecting the two equations in (2.11.6) onto contravariant base vectors.

2.11.4 Matrix formulation redux

The contravariant base vectors $\mathbf{b}^i$ can be arranged at the *columns* of a matrix, $\mathbf{\Phi}$, and the covariant vectors $\widetilde{\mathbf{b}}_i$ can be are arranged at the *columns* of another matrix, $\widetilde{\mathbf{\Phi}}$,

$$\mathbf{\Phi} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{b}^1 & \cdots & \mathbf{b}^N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \qquad \widetilde{\mathbf{\Phi}} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \widetilde{\mathbf{b}}^1 & \cdots & \widetilde{\mathbf{b}}^N \\ \downarrow & \downarrow & \downarrow \end{bmatrix}. \tag{2.11.8}$$

By definition,

$$\widetilde{\mathbf{\Phi}} = \mathbf{\Phi} \cdot \mathbf{R}^{\mathrm{T}}, \qquad \mathbf{\Phi} = \widetilde{\mathbf{\Phi}} \cdot \mathbf{R}^{-\mathrm{T}}, \qquad \mathbf{R} = \widetilde{\mathbf{\Phi}}^{\mathrm{T}} \cdot \mathbf{\Phi}^{-\mathrm{T}}, \tag{2.11.9}$$

where the matrix $\mathbf{R}$ is defined in (2.11.7).

2.11.5 Relation between transformation matrices

Now we recall that

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \boldsymbol{\omega}, \qquad \widetilde{\mathbf{F}}^{\mathrm{T}} \cdot \widetilde{\mathbf{\Phi}} = \widetilde{\boldsymbol{\omega}}, \tag{2.11.10}$$

where $\boldsymbol{\omega}$ is a diagonal matrix whose mth diagonal element is equal to $\omega^{(m)}$, and $\widetilde{\boldsymbol{\omega}}$ is another diagonal matrix whose mth diagonal element is equal to $\widetilde{\omega}^{(m)}$.

The second relation in (2.11.10) can be written as

$$\mathbf{H} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} \cdot \mathbf{R}^{\mathrm{T}} = \widetilde{\boldsymbol{\omega}}, \tag{2.11.11}$$

yielding

$$\mathbf{H} \cdot \boldsymbol{\omega} \cdot \mathbf{R}^{\mathrm{T}} = \widetilde{\boldsymbol{\omega}}, \tag{2.11.12}$$

and then

$$\mathbf{H} = \widetilde{\boldsymbol{\omega}} \cdot \mathbf{R}^{-\mathrm{T}} \cdot \boldsymbol{\omega}^{-1}, \qquad \mathbf{R} = \widetilde{\boldsymbol{\omega}} \cdot \mathbf{H}^{-\mathrm{T}} \cdot \boldsymbol{\omega}^{-1}. \tag{2.11.13}$$

If $\omega^{(j)} = 1$ and $\widetilde{\omega}^{(j)} = 1$ for any j, that is, the two sets of base vectors are orthonormal, then $\mathbf{H} = \mathbf{R}^{-\mathrm{T}}$ and $\mathbf{R} = \mathbf{H}^{-\mathrm{T}}$,

2.11.6 Confirmation by code

The transformation rules between and untilded and tilded bases derived in this section are confirmed by the following Matlab code named *trans1*, located in directory TRANS of TUNLIB. In this implementation, the tilded base arises from the untilded base in terms of a specified matrix, $\mathbf{H}$:

```
%----------
% covariant b
% bcov1 and bcov2
%----------

thcov1 = 0.034*pi;    % arbitrary
thcov2 = 0.334*pi;    % arbitrary

lb1 = 1.4;  % arbitrary
lb2 = 1.8;  % arbitrary

bcov1(1) = lb1*cos(thcov1); bcov1(2) = lb1*sin(thcov1);
bcov2(1) = lb2*cos(thcov2); bcov2(2) = lb2*sin(thcov2);

%---
% contravariant
% bcon1 and bcon2
%
% rotated by 90
%---

thcon1 = thcov2 - 0.5*pi;
thcon2 = thcov1 + 0.5*pi;

lc1 = 2.4; % arbitrary
```

```
lc2 = 1.2; % arbitrary

bcon1(1) = lc1*cos(thcon1); bcon1(2) = lc1*sin(thcon1);
bcon2(1) = lc2*cos(thcon2); bcon2(2) = lc2*sin(thcon2);

omg(1) = bcov1*bcon1';
omg(2) = bcov2*bcon2';

omgmat = [ omg(1),  0.0;
           0.0,     omg(2)];
%---
% matrix F
%---

F = [ bcov1(1), bcov2(1) ;
      bcov1(2), bcov2(2) ];

%---
% matrix H  (arbitrary)
%---

H = [ -1.2  3.3;
       4.3 -1.1];

Hinv = inv(H);

%---
% covariant
% tilded base vectors
%---

bcovt1 = H(1,1)*bcov1 + H(1,2)*bcov2;
bcovt2 = H(2,1)*bcov1 + H(2,2)*bcov2;

%---
% matrix Ftilde
%---

Ft = [ bcovt1(1), bcovt2(1) ;
```

```
       bcovt1(2), bcovt2(2) ];

%---
% contravariant tilde base vectors
% in terms of a matrix
%---

omgtmat = [ 1.9,  0.0;  % arbitrary
           0.0   1.4];

Phit = inv(Ft')*omgtmat;

bcont1(1) = Phit(1,1); bcont1(2) = Phit(2,1);
bcont2(1) = Phit(1,2); bcont2(2) = Phit(2,2);

omgt(1) = bcovt1*bcont1';
omgt(2) = bcovt2*bcont2';

%---
% confirm H and Hinv by (3.7.2)
%---

Hconf = [bcovt1*bcon1'/omg(1) bcovt1*bcon2'/omg(2);
         bcovt2*bcon1'/omg(1) bcovt2*bcon2'/omg(2)];

Hinvconf = [bcov1*bcont1'/omgt(1) bcov1*bcont2'/omgt(2);
            bcov2*bcont1'/omgt(1) bcov2*bcont2'/omgt(2)];

HHconf = [H Hconf]
Hinvconf = [Hinv Hinvconf]

%---
% confirm bcov
%---

bcov1conf = Hinv(1,1)*bcovt1 + Hinv(1,2)*bcovt2;
bcov2conf = Hinv(2,1)*bcovt1 + Hinv(2,2)*bcovt2;

bcovconf = [bcov1 bcov1conf;
```

```
            bcov2 bcov2conf]

%---
% matrix R
%---

R = [bcont1*bcov1'/omg(1) bcont1*bcov2'/omg(2);
     bcont2*bcov1'/omg(1) bcont2*bcov2'/omg(2)];

Rinvconf = [bcon1*bcovt1'/omgt(1) bcon1*bcovt2'/omgt(2);
            bcon2*bcovt1'/omgt(1) bcon2*bcovt2'/omgt(2)];

Rinv = inv(R);

RRinvconf = [Rinv Rinvconf]
```

Running the code generates the following output:

```
HHconf =
   -1.2000    3.3000   -1.2000    3.3000
    4.3000   -1.1000    4.3000   -1.1000
HHinvconf =
    0.0855    0.2564    0.0855    0.2564
    0.3341    0.0932    0.3341    0.0932
bcovconf =
    1.3920    0.1493    1.3920    0.1493
    0.8967    1.5607    0.8967    1.5607
RRinvconf =
   -1.7168    8.3491   -1.7168    8.3491
    3.0351   -1.3730    3.0351   -1.3730
```

The first pair of columns is equal to the second pair of columns, as required.

2.11.7 Cartesian base

If the tilded base vectors constitute a Cartesian base,

$$\widetilde{\mathbf{b}}^i = \widetilde{\mathbf{b}}_i = \mathbf{e}_i \tag{2.11.14}$$

for $i = 1, \ldots, N$, the matrix $\widetilde{\boldsymbol{\omega}}$ is the unit matrix, $\widetilde{\mathbf{F}} = \widetilde{\mathbf{\Phi}}$, and $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}^{-\mathrm{T}}$, which shows that the matrix $\widetilde{\mathbf{F}}$ is orthogonal.

2.11.8 Standard Cartesian base

If the tilded base vectors are the standard Cartesian base vectors,

$$\widetilde{\mathbf{b}}_i = \widetilde{\mathbf{b}}^i = \boldsymbol{\epsilon}_i \tag{2.11.15}$$

for $i = 1, \ldots, N$, the matrix $\widetilde{\boldsymbol{\omega}}$ is the unit matrix, $\widetilde{\mathbf{F}} = \widetilde{\boldsymbol{\Phi}} = \mathbf{I}$, and consequently, $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}$ and $\mathbf{R} = \boldsymbol{\Phi}^{-\mathrm{T}}$. The determinant of the tilde components

Exercise

2.11.1 Run the code *trans1* with a different set of tilde and untilded base vectors.

2.12 Transformation of vector components

A vector can be resolved in terms of covariant or contravariant base vectors in an untilded or tilded base, as discussed in Section 2.7. A relation between the corresponding vector components can be established.

2.12.1 Transformation of contravariant vector components

A vector, $\mathbf{v}$, can be resolved in terms of covariant base vectors as

$$\mathbf{v} = v^i\,\mathbf{b}_i = \widetilde{v}^j\,\widetilde{\mathbf{b}}_j, \tag{2.12.1}$$

where $\widetilde{v}^j$ are the contravariant vector components in the tilded base. Substituting into the first expansion of (2.12.1) the linear expansion $\mathbf{b}_i = H^{-1}_{ij}\widetilde{\mathbf{b}}_j$, we obtain

$$\mathbf{v} = v^i\,H^{-1}_{ij}\widetilde{\mathbf{b}}_j = \widetilde{v}^j\,\widetilde{\mathbf{b}}_j, \tag{2.12.2}$$

which shows that

$$\widetilde{v}^j = H^{-1}_{ij}\,v^i = H^{-\mathrm{T}}_{ji}\,v^i. \tag{2.12.3}$$

Inverting this relation, we obtain

$$v^j = H^{\mathrm{T}}_{ji}\,\widetilde{v}^i = H_{ij}\,\widetilde{v}^i. \tag{2.12.4}$$

We see that the contravariant tilded components arise from the contravariant untilded components, and *vice versa*, by a simple transformation.

2.12.2 Transformation of covariant vector components

A vector, $\mathbf{v}$, may also be resolved in terms of contravariant base vectors as

$$\mathbf{v} = v_i\,\mathbf{b}^i = \widetilde{v}_i\,\widetilde{\mathbf{b}}^i. \tag{2.12.5}$$

Working in a similar fashion, we obtain

$$\widetilde{v}_j = R^{-1}_{ij}\,v_i = R^{-\mathrm{T}}_{ji}\,v_i, \tag{2.12.6}$$

and thus

$$v_j = R^{\mathrm{T}}_{ji}\,v_i = R_{ij}\,v_i. \tag{2.12.7}$$

We see that the covariant tilded components arise from the covariant untilded components, and *vice versa*, by a simple transformation.

2.12.3 Confirmation by code

The following lines of code, continuing code *trans1* listed in Section 2.7, confirm the vector component transformation rules:

```
%---
% arbitrary vector
%---

v = [1.1, -2.1];

%---
% vector components
%---

vcon1 = v*bcon1'/omg(1);
vcon2 = v*bcon2'/omg(2);

vcont1 = v*bcont1'/omgt(1);
```

```
vcont2 = v*bcont2'/omgt(2);

vcov1 = v*bcov1'/omg(1);
vcov2 = v*bcov2'/omg(2);

vcovt1 = v*bcovt1'/omgt(1);
vcovt2 = v*bcovt2'/omgt(2);

%---
% confirm the transformation rules
%---

vvcont = [vcont1, Hinv(1,1)*vcon1+Hinv(2,1)*vcon2;
          vcont2, Hinv(1,2)*vcon1+Hinv(2,2)*vcon2]

vvcon  = [vcon1, H(1,1)*vcont1+H(2,1)*vcont2;
          vcon2, H(1,2)*vcont1+H(2,2)*vcont2]

vvcovt = [vcovt1, Rinv(1,1)*vcov1+Rinv(2,1)*vcov2;
          vcovt2, Rinv(1,2)*vcov1+Rinv(2,2)*vcov2]

vvcov = [vcov1, R(1,1)*vcovt1+R(2,1)*vcovt2;
         vcov2, R(1,2)*vcovt1+R(2,2)*vcovt2]
```

Running the code generates the following output:

```
vvcont =
   -0.3551   -0.3551
    0.3116    0.3116

vvcon =
    1.7658    1.7658
   -1.5144   -1.5144

vvcovt =
   -4.7484   -4.7484
    5.5405    5.5405

vvvov =
    0.4480    0.4480
```

```
   -1.3111   -1.3111
```

The first column is equal to the second column, as required.

2.12.4 Cartesian bases

The transformation rules derived in this section are generalizations of those stated in (1.8.13), for two Cartesian bases, repeated below for convenience,

$$\widetilde{\mathbf{c}} = \mathbf{Q} \cdot \mathbf{c}, \qquad \mathbf{c} = \mathbf{Q}^{\mathrm{T}} \cdot \widetilde{\mathbf{c}}, \tag{2.12.8}$$

where $\mathbf{Q}$ is an orthogonal transformation matrix.

Exercise

2.12.1 Explain how relations (2.12.8) arise from the transformations derived in this section.

2.13 Transformation of tensor components

A tensor, $\mathbf{T}$, can be resolved with respect to its pure contravariant components in an untilded or tilded base as

$$\mathbf{T} = T^{ij}\, \mathbf{b}_i \otimes \mathbf{b}_j = \widetilde{T}^{pq}\, \widetilde{\mathbf{b}}_p \otimes \widetilde{\mathbf{b}}_q, \tag{2.13.1}$$

where summation is implied over the repeated indices, i, j, p, and q.

2.13.1 Transformation of contravariant tensor components

Substituting into the first expression in (2.13.1) the expansions

$$\mathbf{b}_i = H^{-1}_{ip} \widetilde{\mathbf{b}}_p, \qquad \mathbf{b}_j = H^{-1}_{jq} \widetilde{\mathbf{b}}_q, \tag{2.13.2}$$

we obtain

$$\mathbf{T} = T^{ij}\, H^{-1}_{ip}\, H^{-1}_{jq}\, \widetilde{\mathbf{b}}_p \otimes \widetilde{\mathbf{b}}_q = \widetilde{T}^{pq}\, \widetilde{\mathbf{b}}_p \otimes \widetilde{\mathbf{b}}_q, \tag{2.13.3}$$

which shows that

$$\widetilde{T}^{pq} = H^{-\mathrm{T}}_{pi}\, T^{ij}\, H^{-1}_{jq}, \tag{2.13.4}$$

where $-\mathrm{T}$ denotes the transpose of the inverse, which is equal to the inverse of the transpose. Inverting this relation, we obtain

$$T^{pq} = H^{\mathrm{T}}_{pi} \widetilde{T}^{ij} H_{jq}. \tag{2.13.5}$$

We see that the contravariant tilded components arise from the contravariant untilded components, and *vice versa*, by a simple transformation.

2.13.2 Transformation of covariant tensor components

Working in a similar fashion, we find that

$$\widetilde{T}_{pq} = R^{-\mathrm{T}}_{pi} T_{ij} R^{-1}_{jq}, \qquad T_{pq} = R^{\mathrm{T}}_{pi} \widetilde{T}_{ij} R_{jq}. \tag{2.13.6}$$

These relations are the same as those shown in (2.13.4) and (2.13.5) with H replaced by R.

2.13.3 Transformation of mixed tensor components

Working in a similar fashion, we find the following transformation rules for mixed tensor components,

$$\widetilde{T}^{p}_{\circ q} = H^{-\mathrm{T}}_{pi} T^{i}_{\circ j} R^{-1}_{jq}, \qquad T^{p}_{\circ q} = H^{\mathrm{T}}_{pi} \widetilde{T}^{i}_{\circ j} R_{jq} \tag{2.13.7}$$

and

$$\widetilde{T}_{p}^{\circ q} = R^{-\mathrm{T}}_{pi} T_{i}^{\circ j} H^{-1}_{jq}, \qquad T_{p}^{\circ q} = R^{-1}_{pi} \widetilde{T}_{i}^{\circ j} H_{jq}. \tag{2.13.8}$$

These relations are the same as those shown in (2.13.4) and (2.13.5) with H partially replaced by R.

2.13.4 Determinants

Taking the determinant of (2.13.4), and recalling that the determinant of a matrix inverse is equal to the inverse of the determinant, we obtain

$$\det[\widetilde{T}^{pq}] = \frac{1}{\det^2(\mathbf{H})} \det[T^{ij}]. \tag{2.13.9}$$

Taking the determinant of the first relation in (2.13.6), we obtain the counterpart of (2.13.9) ,

$$\det[\widetilde{T}_{pq}] = \frac{1}{\det^2(\mathbf{R})} \det[T_{ij}]. \tag{2.13.10}$$

Taking the determinants of the first relations in (2.13.7) and (2.13.8), we find that

$$\det[\widetilde{T}^{p}_{\circ q}] = \frac{1}{\det(\mathbf{R})\,\det(\mathbf{H})}\,\det[T^{p}_{\circ q}] \qquad (2.13.11)$$

and

$$\det[\widetilde{T}^{\circ q}_{p}] = \frac{1}{\det(\mathbf{R})\,\det(\mathbf{H})}\,\det[T^{\circ q}_{p}]. \qquad (2.13.12)$$

These determinants should not be confused with that of the tensor, $\mathbf{T}$.

2.13.5 Confirmation by code

The following Matlab code named *trans2*, located in directory TRANS of TUNLIB, confirms these transformation rules:

```
%----------
% bcov1 and bcov2 (covariant base vectors)
%----------

thbcov1 = 0.034*pi;    % arbitrary
thbcov2 = 0.334*pi;    % arbitrary

lb1 = 1.4;  % arbitrary
lb2 = 1.8;  % arbitrary

bcov1(1) = lb1*cos(thbcov1); bcov1(2) = lb1*sin(thbcov1);
bcov2(1) = lb2*cos(thbcov2); bcov2(2) = lb2*sin(thbcov2);

%---
% bcon1 and bcon2  (contravariant base vector)
%---

thbcon1 = thbcov2 - 0.5*pi;
thbcon2 = thbcov1 + 0.5*pi;

lc1 = 2.4; % arbitrary
lc2 = 1.2; % arbitrary

bcon1(1) = lc1*cos(thbcon1); bcon1(2) = lc1*sin(thbcon1);
```

```
bcon2(1) = lc2*cos(thbcon2); bcon2(2) = lc2*sin(thbcon2);

%---
% projections
%---

omg(1) = bcov1*bcon1';
omg(2) = bcov2*bcon2';

%---
% tensor T
%---

T = [ 1 2;   % arbitrary
      3 4];
%---
% con-cov components
%---

Tnv(1,1) = bcon1*T*bcov1'/(omg(1)*omg(1));
Tnv(1,2) = bcon1*T*bcov2'/(omg(1)*omg(2));
Tnv(2,1) = bcon2*T*bcov1'/(omg(2)*omg(1));
Tnv(2,2) = bcon2*T*bcov2'/(omg(2)*omg(2));

%---
% cov-con components
%---

Tvn(1,1) = bcov1*T*bcon1'/(omg(1)*omg(1));
Tvn(1,2) = bcov1*T*bcon2'/(omg(1)*omg(2));
Tvn(2,1) = bcov2*T*bcon1'/(omg(2)*omg(1));
Tvn(2,2) = bcov2*T*bcon2'/(omg(2)*omg(2));

%---
% cov-cov components
%---

Tvv(1,1) = bcov1*T*bcov1'/(omg(1)*omg(1));
Tvv(1,2) = bcov1*T*bcov2'/(omg(1)*omg(2));
```

```
Tvv(2,1) = bcov2*T*bcov1'/(omg(2)*omg(1));
Tvv(2,2) = bcov2*T*bcov2'/(omg(2)*omg(2));

%---
% con-con components
%---

Tnn(1,1) = bcon1*T*bcon1'/(omg(1)*omg(1));
Tnn(1,2) = bcon1*T*bcon2'/(omg(1)*omg(2));
Tnn(2,1) = bcon2*T*bcon1'/(omg(2)*omg(1));
Tnn(2,2) = bcon2*T*bcon2'/(omg(2)*omg(2));

%---
% matrix H
%---

H = [ -1.2  3.3;   % arbitrary
       4.3 -1.1];

Hinv = inv(H);

%---
% tilde base vectors
%---

bcovt1 = H(1,1)*bcov1 + H(1,2)*bcov2;
bcovt2 = H(2,1)*bcov1 + H(2,2)*bcov2;

%---
% contravariant tilde base vectors
% computed in terms of a matrix
%---

AMAT = [ bcovt1(1), bcovt2(1);
         bcovt1(2), bcovt2(2)];

omgtMAT = [ 1.9,  0.0;  % arbitrary
            0.0   1.4];
```

```
BMAT = inv(AMAT')*omgtMAT;

bcont1(1) = BMAT(1,1); bcont1(2) = BMAT(2,1);
bcont2(1) = BMAT(1,2); bcont2(2) = BMAT(2,2);

omgt(1) = bcovt1*bcont1';
omgt(2) = bcovt2*bcont2';

%---
% matrix R
%---

R = [bcont1*bcov1'/omg(1) bcont1*bcov2'/omg(2);
     bcont2*bcov1'/omg(1) bcont2*bcov2'/omg(2)];

Rinv = inv(R);

%---
% con-cov components
%---

Tnvt(1,1) = bcont1*T*bcovt1'/(omgt(1)*omgt(1));
Tnvt(1,2) = bcont1*T*bcovt2'/(omgt(1)*omgt(2));
Tnvt(2,1) = bcont2*T*bcovt1'/(omgt(2)*omgt(1));
Tnvt(2,2) = bcont2*T*bcovt2'/(omgt(2)*omgt(2));

%---
% cov-con components
%---

Tvnt(1,1) = bcovt1*T*bcont1'/(omgt(1)*omgt(1));
Tvnt(1,2) = bcovt1*T*bcont2'/(omgt(1)*omgt(2));
Tvnt(2,1) = bcovt2*T*bcont1'/(omgt(2)*omgt(1));
Tvnt(2,2) = bcovt2*T*bcont2'/(omgt(2)*omgt(2));

%---
% cov-cov components
%---
```

```
Tvvt(1,1) = bcovt1*T*bcovt1'/(omgt(1)*omgt(1));
Tvvt(1,2) = bcovt1*T*bcovt2'/(omgt(1)*omgt(2));
Tvvt(2,1) = bcovt2*T*bcovt1'/(omgt(2)*omgt(1));
Tvvt(2,2) = bcovt2*T*bcovt2'/(omgt(2)*omgt(2));

%---
% con-con components
%---

Tnnt(1,1) = bcont1*T*bcont1'/(omgt(1)*omgt(1));
Tnnt(1,2) = bcont1*T*bcont2'/(omgt(1)*omgt(2));
Tnnt(2,1) = bcont2*T*bcont1'/(omgt(2)*omgt(1));
Tnnt(2,2) = bcont2*T*bcont2'/(omgt(2)*omgt(2));

%---
% confirm
%---

Tconf = [Tnvt Hinv'*Tnv*Rinv;
         Tvnt Rinv'*Tvn*Hinv;
         Tvvt Rinv'*Tvv*Rinv;
         Tnnt Hinv'*Tnn*Hinv]
%---
% confirm determinants
%---

detH = det(H);
detR = det(R);

Detconf = [det(Tnvt) det(Tnv)/detR/detH;
           det(Tvnt) det(Tvn)/detR/detH;
           det(Tvvt) det(Tvv)/detR/detR;
           det(Tnnt) det(Tnn)/detH/detH]
```

Running the code generates the following output:

```
Tconf =
    2.6240    1.5393    2.6240    1.5393
    0.5060    0.0103    0.5060    0.0103
    2.6020    1.0350    2.6020    1.0350
```

```
    0.8275    0.0402    0.8275    0.0402
   36.7177   21.3738   36.7177   21.3738
   11.5097    1.4001   11.5097    1.4001
    0.1859    0.0746    0.1859    0.0746
    0.0365   -0.0010    0.0365   -0.0010

Detconf =
   -0.7519   -0.7519
   -0.7519   -0.7519
 -194.5988 -194.5988
   -0.0029   -0.0029
```

As expected, the first pair of columns is identical to the second pair.

2.13.6 Determinant of a tensor

If the tilded base vectors constitute a Cartesian base,

$$\widetilde{\mathbf{b}}^i = \widetilde{\mathbf{b}}_i = \mathbf{e}_i \tag{2.13.13}$$

for $i = 1, \ldots, N$, the matrices $\widetilde{\mathbf{F}} = \widetilde{\mathbf{\Phi}}$ are orthogonal with unit determinant and $\widetilde{\boldsymbol{\omega}}$ is the identity matrix. From the third equations in (2.11.9) and (2.11.4), we find that

$$\det(\mathbf{H}) = \frac{1}{\det(\mathbf{F})} \equiv \frac{1}{\mathcal{J}_\circ}, \tag{2.13.14}$$

and

$$\det(\mathbf{R}) = \frac{1}{\det(\mathbf{\Phi})} \equiv \frac{1}{\mathcal{J}^\circ}, \tag{2.13.15}$$

as shown in (2.2.10). The determinant of the tilde components in the expressions derived in Section 2.5 is equal to the tensor determinant,

$$\det(\mathbf{T}) = \mathcal{J}_\circ^2 \det[T^{ij}] = \mathcal{J}^{\circ 2} \det[T_{ij}] \tag{2.13.16}$$

and

$$\det(\mathbf{T}) = \det(\boldsymbol{\omega}) \det[T^p_{\circ q}] = \det(\boldsymbol{\omega}) \det[T_p^{\circ q}]. \tag{2.13.17}$$

Exercise

2.13.1 Confirm by computation that, if the second base vector are the standard Cartesian vectors, $\widetilde{\mathbf{b}}_i = \boldsymbol{\epsilon}_i$, and $\widetilde{\mathbf{b}}^i = \boldsymbol{\epsilon}_i$, the determinants of the tilde components is equal to the matrix determinant, $\det(\mathbf{T})$.

2.14 High-order tensors

Biorthogonal bases can be employed to describe third- and higher-order indexed tensors.

For a three-index tensor, we may write

$$\mathbf{T} = T^{ijk}\,\mathbf{b}_i \otimes \mathbf{b}_j \otimes \mathbf{b}_k, \tag{2.14.1}$$

and also

$$\mathbf{T} = T_{ijk}\,\mathbf{b}^i \otimes \mathbf{b}^j \otimes \mathbf{b}^k, \tag{2.14.2}$$

where T^{ijk} are pure contravariant components, T_{ijk} are pure covariant components, and summation is implied over the repeated indices, i, j, and k.

Moreover, we may introduce expansions in mixed components, such as the expansion

$$\mathbf{T} = T_{i\circ k}^{\circ j}\,\mathbf{b}^i \otimes \mathbf{b}_j \otimes \mathbf{b}^k, \tag{2.14.3}$$

in this and other combinations of tensor components.

2.14.1 Component conversion

Contravariant, covariant, and mixed tensor components can be deduced from one another, as discussed in Section 2.8 for two-index components.

For example, multiplying (2.14.2) and (2.14.3) from the left with $\mathbf{b}_m$ and from the right with $\mathbf{b}_n$, we find that

$$\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n = T_{mjn}\,\omega^{(m)}\,\omega^{(n)}\,\mathbf{b}^j = T_{m\circ n}^{\circ j}\,\omega^{(m)}\,\omega^{(n)}\,\mathbf{b}_j, \tag{2.14.4}$$

where m and n are free indices. Now multiplying through this expression by $\mathbf{b}_p$, where p is a free index, we obtain

$$\mathbf{b}_p \cdot (\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n) = T_{mpn}\, \omega^{(m)}\, \omega^{(n)}\, \omega^{(p)} = T_{m \circ k}^{\circ j}\, \omega^{(m)}\, \omega^{(n)}\, b_{jp}, \tag{2.14.5}$$

and then

$$T_{mpn} = \frac{1}{\omega^{(p)}}\, T_{m \circ k}^{\circ j}\, b_{jp}, \tag{2.14.6}$$

which implements the usual rule for lowering an index.

From the first equality in (2.14.5), we find that

$$T_{mpn} = \frac{1}{\omega^{(m)}\, \omega^{(n)}\, \omega^{(p)}}\, \mathbf{b}_p \cdot (\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n), \tag{2.14.7}$$

where

$$\mathbf{b}_p \cdot (\mathbf{b}_m \cdot \mathbf{T} \cdot \mathbf{b}_n) = (\mathbf{b}_p)_j\, \omega^{(j)} (\mathbf{b}_m)_i\, \omega^{(i)} (\mathbf{b}_n)_k\, \omega^{(k)}\, T^{ijk}, \tag{2.14.8}$$

and summation is implied over the repeated indices, i, j, and k.

Exercises

2.14.1 Enumerate possible expansion combinations of three-index tensors.

2.14.2 Derive tensor transformation rules for the purely contravariant components of a three-index tensor.

2.15 Alternating tensor

We restrict our attention to three dimensions, $N = 3$, and construct the alternating tensor from the representations

$$\begin{aligned} \boldsymbol{\xi} = \xi^{ijk}\, \mathbf{b}_i \otimes \mathbf{b}_j \otimes \mathbf{b}_k &= \xi_i^{\circ jk}\, \mathbf{b}^i \otimes \mathbf{b}_j \otimes \mathbf{b}_k \\ &= \cdots = \xi_{ijk}\, \mathbf{b}^i \otimes \mathbf{b}^j \otimes \mathbf{b}^k, \end{aligned} \tag{2.15.1}$$

where $\xi^{ijk}, \ldots, \xi_{ijk}$ are tensor components.

2.15.1 Contravariant and covariant components

We will confirm that the contravariant and covariant components of the alternating tensor are given by

$$\xi^{ijk} = \frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk}, \qquad \xi_{ijk} = \frac{1}{\mathcal{J}^\circ}\,\epsilon_{ijk}, \tag{2.15.2}$$

in agreement with the representation (1.16.7) in terms of an arbitrary trio of vectors, where ϵ_{ijk} is the Levi–Civita symbol,

$$\mathcal{J}_\circ = [\,\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\,], \qquad \mathcal{J}^\circ = [\,\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3\,] \tag{2.15.3}$$

are the covariant and contravariant Jacobian metrics satisfying

$$\mathcal{J}_\circ \mathcal{J}^\circ = \det(\boldsymbol{\omega}) \tag{2.15.4}$$

and

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] \equiv (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \tag{2.15.5}$$

is the triple mixed product representing the volume of the parallelepiped whose edges are three arbitrary vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Cyclic permutation of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, preserves the triple mixed product; non-cyclic permutation preserves the magnitude but changes the sign.

Consequently,

$$\boldsymbol{\xi} = \frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk}\,\mathbf{b}_i \otimes \mathbf{b}_j \otimes \mathbf{b}_k = \frac{1}{\mathcal{J}^\circ}\,\epsilon_{ijk}\,\mathbf{b}^i \otimes \mathbf{b}^j \otimes \mathbf{b}^k. \tag{2.15.6}$$

Projecting this expansion from the left with $\mathbf{b}_p$, from the right with $\mathbf{b}_q$, and then with $\mathbf{b}_n$, we obtain

$$\begin{aligned}(\mathbf{b}_p) \cdot (\mathbf{b}_m \cdot \boldsymbol{\xi} \cdot \mathbf{b}_n) &= \frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk}\,b_{pi}\,b_{qj}\,b_{nk} \\ &= \frac{1}{\mathcal{J}^\circ}\,\epsilon_{ijk}\,\omega^{(p)}\,\omega^{(q)}\,\omega^{(n)}\,\delta_{pi}\,\delta_{qj}\,\delta_{nk},\end{aligned} \tag{2.15.7}$$

and therefore

$$\mathcal{J}^\circ \epsilon_{ijk}\,b_{pi}\,b_{qj}\,b_{nk} = \mathcal{J}_\circ\,\epsilon_{pqn}\,\omega^{(p)}\,\omega^{(q)}\,\omega^{(n)}. \tag{2.15.8}$$

Rearranging, we find that

$$\epsilon_{pqn} = c\,\epsilon_{ijk}\,b_{ip}\,b_{jq}\,b_{kn}, \tag{2.15.9}$$

where

$$c = \frac{\mathcal{J}^\circ}{\mathcal{J}_\circ}\frac{1}{\det(\boldsymbol{\omega})} = \frac{1}{\mathcal{J}_\circ^2} = \frac{1}{\det(\mathbf{b})}, \tag{2.15.10}$$

which is an identity, as shown in (2.4.17).

2.15.2 *Cross product of two vectors*

The cross product of two vectors, $\mathbf{v}$ and $\mathbf{u}$, is given by

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = \boldsymbol{\xi} : (\mathbf{v} \otimes \mathbf{u}). \tag{2.15.11}$$

We find that

$$\mathbf{w} = (\frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk}\,\mathbf{b}_i \otimes \mathbf{b}_j \otimes \mathbf{b}_k) : (v_p u_q\,\mathbf{b}^p \otimes \mathbf{b}^q), \tag{2.15.12}$$

which can be restated as

$$\mathbf{w} = \frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk} v_p u_q\,(\mathbf{b}_i \otimes \mathbf{b}_j \otimes \mathbf{b}_k) : (\mathbf{b}^p \otimes \mathbf{b}^q), \tag{2.15.13}$$

and then

$$\mathbf{w} = \frac{1}{\mathcal{J}_\circ}\,\epsilon_{ijk}\,v_j u_k\,\omega^{(j)}\omega^{(k)}\,\mathbf{b}_i = \frac{1}{\mathcal{J}_\circ}\det(\boldsymbol{\omega})\,\epsilon_{ijk}\,v_j u_k\,\frac{1}{\omega^{(i)}}\,\mathbf{b}_i. \tag{2.15.14}$$

Simplifying, we obtain

$$\mathbf{w} = \mathcal{J}^\circ\,\epsilon_{ijk}\,v_j u_k\,\frac{1}{\omega^{(i)}}\,\mathbf{b}_i, \tag{2.15.15}$$

which reproduces the first expression in (2.4.12).

Exercise

2.15.1 Prove the derivation shown in (2.15.14).

Chapter 3

Introduction to non-Cartesian coordinates

Non-Cartesian, rectilinear or curvilinear coordinates are employed to accommodate the geometry of a particular domain of interest in solving partial differential equations equations by analytical or numerical methods. The main reason for using such coordinates in science and engineering applications is to facilitate the implementation in boundary conditions.

For example, if a solution is sought inside a sphere of radius a, then it is desirable to use spherical polar coordinates so that the boundary of the solution domain is described by $r = a$, where r is the distance from the origin. If the solution domain is a channel with wavy walls, pertinent boundary-fitted coordinates are employed.

In this chapter, we illustrate fundamental notions and concepts underlying the construction and usage of curvilinear coordinates in two dimensions. The discussion is an extension of that presented Chapter 2 on biorthogonal vector and tensor bases. Following the derivation of fundamental definitions and expressions, we will develop and implement finite-difference methods for solving the Laplace and Poisson equations. Other linear differential equations can be solved by similar methods. The discussion will be extended, completed, and formalized in Chapter 4.

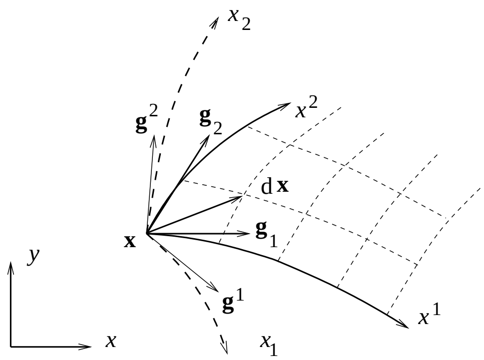

FIGURE 3.1.1 Illustration of curvilinear coordinates in a plane: $(\mathbf{g}_1, \mathbf{g}_2)$ are the covariant base vectors and $(\mathbf{g}^1, \mathbf{g}^2)$ are the contravariant base vectors. Correspondingly, (x^1, x^2) are the contravariant coordinates and (x_1, x_2) are the covariant coordinates.

3.1 Covariant base vectors and contravariant coordinates

Consider two continuous intersecting families of generally curved lines in the xy plane, parametrized by two variables, (x^1, x^2), called contravariant coordinates, as shown in Figure 3.1.1. For reasons that will become evident in hindsight, the indices were written intentionally as superscripts instead of subscripts. To be clear, the dashed coordinate lines shown in Figure 3.1.1 are constructed by holding x^1 or x^2 constant.

3.1.1 Position

The Cartesian components of the position, $\mathbf{x} = (x, y)$, can be regarded as functions of x^1 and x^2,

$$\mathbf{x}(x^1, x^2). \tag{3.1.1}$$

For example, we may denote for convenience $\xi = x^1$ and $\eta = x^2$, and consider contravariant coordinates lines described by

$$x = 2\pi(\xi + \alpha\,\eta^2)\,L,$$
$$y = (1 - \beta\,\xi\eta)\,\cos(2\pi\xi)\,L + \gamma\eta\,L, \tag{3.1.2}$$

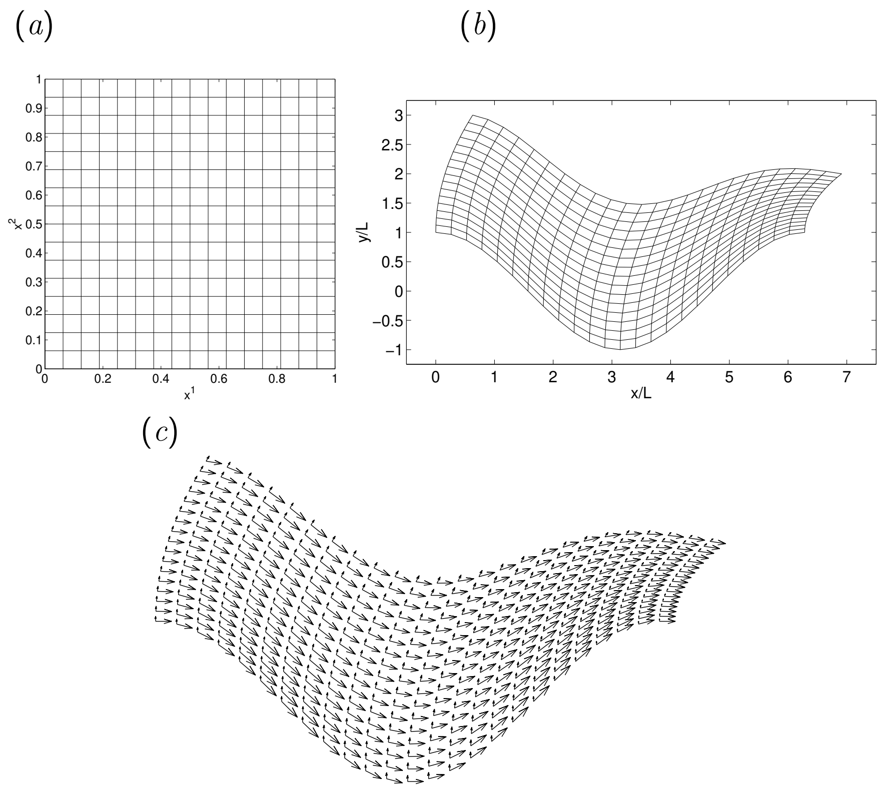

FIGURE 3.1.2 Contravariant coordinate lines described by equations (3.1.2) with $\alpha = 0.1$, $\beta = 1$, and $\gamma = 2$, for $0 \leq x^1 < 1$ and $0 \leq x^2 < 1$. (a) Cartesian lines in a parametric square, (b) corresponding coordinate lines in the xy plane, and (c) covariant base vector field.

where ξ, η are dimensionless curvilinear coordinates, L is a fixed length, and α, β, γ are dimensionless constants Coordinate lines for $0 \leq \xi < 1$ and $0 \leq \eta < 1$ generated using these expressions for $\alpha = 0.1$, $\beta = 1.0$, $\gamma = 2.0$ are shown in Figure 3.1.2.

3.1.2 Covariant base vectors

Next, we introduce a pair of tangential base vectors, named the *covariant* base vectors, defined as partial derivatives of the position vector,

$$\mathbf{g}_1 \equiv \frac{\partial \mathbf{x}}{\partial x^1}, \qquad \mathbf{g}_2 \equiv \frac{\partial \mathbf{x}}{\partial x^2}, \tag{3.1.3}$$

as shown in Figure 3.1.1. The derivative with respect to x^1 is taken under constant x^2 and the derivative with respect to x^2 is taken under constant x^1. In Cartesian component form,

$$\mathbf{g}_1 = \frac{\partial x}{\partial x^1}\,\mathbf{e}_x + \frac{\partial y}{\partial x^1}\,\mathbf{e}_y, \qquad \mathbf{g}_2 = \frac{\partial x}{\partial x^2}\,\mathbf{e}_x + \frac{\partial y}{\partial x^2}\,\mathbf{e}_y, \tag{3.1.4}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along the x and y axes. The covariant base vectors, $\mathbf{g}_1$ and $\mathbf{g}_2$, are unit vectors only if the coordinates x^1 and x^2 measure physical arc length in their respective directions.

For the coordinates described in (3.1.2), we find by straightforward differentiation that

$$\begin{aligned}
\mathbf{g}_1 &= \frac{\partial \mathbf{x}}{\partial \xi} = 2\pi\,\mathbf{e}_x + \big(-\beta\eta\,\cos(2\pi\xi) - 2\pi(1-\beta\,\xi\eta)\,\sin(2\pi\xi)\big)\,\mathbf{e}_y, \\
\mathbf{g}_2 &= \frac{\partial \mathbf{x}}{\partial \eta} = 4\pi\alpha\eta\,\mathbf{e}_x + \big(-\beta\,\xi\,\cos(2\pi\xi) + \gamma\big)\,\mathbf{e}_y.
\end{aligned} \tag{3.1.5}$$

In this case, because the parameters ξ and η do not express physical arc length along corresponding lines, $\mathbf{g}_1$ and $\mathbf{g}_2$ are not unit vectors.

3.1.3 Orthogonal coordinates

The ordered doublet (x^1, x^2) comprises a pair of generally non-orthogonal curvilinear coordinates with associated covariant base vectors $(\mathbf{g}_1, \mathbf{g}_2)$. If the lines of constant x^1 and x^2 are straight, the coordinates are rectilinear. If the angle subtended between the two vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ is equal to 90° at every point in the xy plane, we obtain orthogonal rectilinear or curvilinear coordinates.

3.1.4 Differential displacement

An infinitesimal displacement vector in the xy plane at a point, $\mathbf{x}$, can be expressed as

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial \mathbf{x}}{\partial x^2}\,\mathrm{d}x^2, \tag{3.1.6}$$

where $\mathrm{d}x^1$ and $\mathrm{d}x^2$ are differential increments regarded as contravariant components of $\mathrm{d}\mathbf{x}$. In terms of the covariant base vectors as

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1\,\mathrm{d}x^1 + \mathbf{g}_2\,\mathrm{d}x^2. \tag{3.1.7}$$

Given expressions for the covariant base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ in terms of x^1 and x^2, equation (3.1.7) can be integrated analytically or numerically to provide us with an explicit expression for the position vector, $\mathbf{x}(x^1, x^2)$.

3.1.5 Covariant metric coefficients

It will be useful to introduce the *covariant components of the metric tensor*, also called the *covariant metric coefficients*, defined as

$$g_{11} \equiv \mathbf{g}_1 \cdot \mathbf{g}_1 = |\mathbf{g}_1|^2, \qquad g_{22} \equiv \mathbf{g}_2 \cdot \mathbf{g}_2 = |\mathbf{g}_2|^2,$$
$$g_{12} = g_{21} \equiv \mathbf{g}_1 \cdot \mathbf{g}_2 = \mathbf{g}_2 \cdot \mathbf{g}_1. \tag{3.1.8}$$

These coefficients can be collected into a symmetric matrix of covariant metric coefficients, denoted by

$$\mathbf{g} \equiv \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \tag{3.1.9}$$

The determinant of this matrix is

$$g \equiv \det(\mathbf{g}) = g_{11}g_{22} - g_{12}^2. \tag{3.1.10}$$

If $\mathbf{g}_1$ and $\mathbf{g}_2$ are unit vectors, $g_{11} = 1$ and $g_{22} = 1$.

If the coordinates are orthogonal, $g_{12} = 0$ and $g_{21} = 0$. More generally, we denote by θ the angle between $\mathbf{g}_1$ and $\mathbf{g}_2$, and use the geometrical interpretation of the dot product to write

$$g_{12} \equiv \mathbf{g}_1 \cdot \mathbf{g}_2 = |\mathbf{g}_1||\mathbf{g}_2|\,\cos\theta. \tag{3.1.11}$$

Substituting this expression into (3.1.10), we obtain

$$g = |\mathbf{g}_1|^2|\mathbf{g}_2|^2(1 - \cos^2\theta) = |\mathbf{g}_1|^2|\mathbf{g}_2|^2\sin^2\theta, \tag{3.1.12}$$

yielding

$$g = |\mathbf{g}_1 \times \mathbf{g}_2|^2. \tag{3.1.13}$$

We see that $\sqrt{g}$ is the area of a parallelogram whose sides are defined by the base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$.

3.1.6 Fundamental form of a plane

Using the definitions of the covariant metric coefficients, we find that the square of the magnitude of a differential displacement is given by

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = (\mathbf{g}_i \, \mathrm{d}x^i) \cdot (\mathbf{g}_j \, \mathrm{d}x^j), \tag{3.1.14}$$

where summation is implied over the repeated indices, i and j. Carrying out the multiplications and invoking the definition of the covariant metric coefficients, we find that

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = g_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j. \tag{3.1.15}$$

The expansion given in (3.1.15) constitutes the fundamental form of a plane.

3.1.7 Extracting differential coordinates

In the general case of non-orthogonal curvilinear coordinates where the variables x^1 and x^2 do not measure physical arc length from a designated origin,

$$\mathbf{g}_1 \cdot \mathbf{g}_2 \neq 0, \qquad \mathbf{g}_1 \cdot \mathbf{g}_1 \neq 1, \qquad \mathbf{g}_2 \cdot \mathbf{g}_2 \neq 1. \tag{3.1.16}$$

The first inequality expressing non-orthogonality prevents us from computing the differential components $\mathrm{d}x^1$ and $\mathrm{d}x^2$ in (3.1.7) by projecting the differential of the position vector given in (3.1.7) onto each base vector, that is,

$$\mathrm{d}x^1 \neq \mathrm{d}\mathbf{x} \cdot \mathbf{g}_1 \, \frac{1}{\mathbf{g}_1 \cdot \mathbf{g}_1}, \qquad \mathrm{d}x^2 \neq \mathrm{d}\mathbf{x} \cdot \mathbf{g}_2 \, \frac{1}{\mathbf{g}_2 \cdot \mathbf{g}_2}. \tag{3.1.17}$$

Instead, the differential components must be found by solving a system of linear two equations for two unknowns originating by projecting equation (3.1.7) onto $\mathbf{g}_1$ or $\mathbf{g}_2$,

$$\begin{aligned} g_{11} \, \mathrm{d}x^1 + g_{12} \, \mathrm{d}x^2 &= \mathrm{d}\mathbf{x} \cdot \mathbf{g}_1, \\ g_{21} \, \mathrm{d}x^1 + g_{22} \, \mathrm{d}x^2 &= \mathrm{d}\mathbf{x} \cdot \mathbf{g}_2. \end{aligned} \tag{3.1.18}$$

Using Cramer's rule, we find that

$$\mathrm{d}x^1 = \frac{1}{g}\,\mathrm{d}\mathbf{x}\cdot\big(g_{22}\,\mathbf{g}_1 - g_{12}\,\mathbf{g}_2\big),$$
$$\mathrm{d}x^2 = \frac{1}{g}\,\mathrm{d}\mathbf{x}\cdot\big(-g_{12}\,\mathbf{g}_1 + g_{11}\,\mathbf{g}_2\big), \qquad (3.1.19)$$

where $g = g_{11}g_{22} - g_{12}^2$, as given in (3.1.10).

3.1.8 Contravariant base vectors

The right-hand sides of the two expressions in (3.1.19) motivate introducing a pair of base vectors,

$$\mathbf{g}^1 \equiv \frac{1}{g}\big(g_{22}\,\mathbf{g}_1 - g_{12}\,\mathbf{g}_2\big),$$
$$\mathbf{g}^2 \equiv \frac{1}{g}\big(-g_{12}\,\mathbf{g}_1 + g_{11}\,\mathbf{g}_2\big), \qquad (3.1.20)$$

called the *contravariant base vectors*. Equations (3.1.19) may then be expressed in the compact form

$$\mathrm{d}x^1 = \mathrm{d}\mathbf{x}\cdot\mathbf{g}^1, \qquad \mathrm{d}x^2 = \mathrm{d}\mathbf{x}\cdot\mathbf{g}^2. \qquad (3.1.21)$$

According to these formulas, $\mathrm{d}x^1$ and $\mathrm{d}x^2$ can be extracted from $\mathrm{d}\mathbf{x}$ merely by two projections.

3.1.9 Covariant from contravariant base vectors

Conversely, the covariant base vectors arise from the contravariant base vectors as

$$\mathbf{g}_1 = g_{11}\,\mathbf{g}^1 + g_{12}\,\mathbf{g}^2, \qquad \mathbf{g}_2 = g_{12}\,\mathbf{g}^1 + g_{22}\,\mathbf{g}^2, \qquad (3.1.22)$$

which provide us with formulas for lowering the indices,

$$\mathbf{g}_i = g_{ij}\mathbf{g}^j. \qquad (3.1.23)$$

In the case of orthogonal coordinates, $g_{12} = 0$, the base vector $\mathbf{g}^1$ is parallel to $\mathbf{g}_1$ and the base vector $\mathbf{g}^2$ is parallel to $\mathbf{g}_2$.

3.1.10 Biorthonormality

Using the definitions (3.1.20), we find that

$$\mathbf{g}^1 \cdot \mathbf{g}_1 = 1, \quad \mathbf{g}^2 \cdot \mathbf{g}_1 = 0, \quad \mathbf{g}^1 \cdot \mathbf{g}_2 = 0, \quad \mathbf{g}^2 \cdot \mathbf{g}_2 = 1, \quad (3.1.24)$$

which reveals that $\mathbf{g}^2$ is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}^1$ is orthogonal to $\mathbf{g}_2$, as shown in Figure 3.1.1. We have thus arrived at the biorthonormality condition

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta_{ij}, \quad (3.1.25)$$

where δ_{ij} is Kronecker's delta. The covariant base vectors, $\mathbf{g}^i$, and the contravariant base vectors, $\mathbf{g}_j$, constitute a biorthonormal set.

3.1.11 Confirmation by code

a Matlab code named *nonortho*, located in directory NONORTHO of TUNLIB, defines a pair of arbitrary covariant base vectors and computes the associated contravariant base vectors. In the numerical implementation, the contravariant base vectors are computed as rotations of the covariant base vectors by $\pi/2$, followed by normalization:

```
%----------
% define two arbitrary covariant base vectors
% gcov1 and gcov2
%----------

ancov1 = 0.034*pi;    % arbitrary angle
ancov2 = 0.334*pi;    % arbitrary angle

lcov1 = 1.4;  % arbitrary length
lcov2 = 1.8;  % arbitrary length

gcov1(1) = lcov1*cos(ancov1); gcov1(2) = lcov1*sin(ancov1);
gcov2(1) = lcov2*cos(ancov2); gcov2(2) = lcov2*sin(ancov2);

%---
% covariant components of the metric tensor
%---
```

```
covmet(1,1) = gcov1(1)*gcov1(1) + gcov1(2)*gcov1(2);
covmet(1,2) = gcov1(1)*gcov2(1) + gcov1(2)*gcov2(2);
covmet(2,1) = covmetric(1,2);
covmet(2,2) = gcov2(1)*gcov2(1) + gcov2(2)*gcov2(2);

covg = det(covmetric);  % determinant

%---
% compute the associated contravariant base vectors
% gcon1 and gcon2
%---

ancon1 = ancov2 - 0.5*pi;
ancon2 = ancov1 + 0.5*pi;

lcon1 = 2.4; % arbitrary
lcon2 = 1.2; % arbitrary

gcon1(1) = lcon1*cos(ancon1); gcon1(2) = lcon1*sin(ancon1);
gcon2(1) = lcon2*cos(ancon2); gcon2(2) = lcon2*sin(ancon2);

%---
% normalize the contravariant base vectors
% so that cov1*con1 = 1,  cov2*con2 = 1,
%---

norm1 = gcov1(1)*gcon1(1) + gcov1(2)*gcon1(2);
norm2 = gcov2(1)*gcon2(1) + gcov2(2)*gcon2(2);

gcon1(1) = gcon1(1)/norm1;
gcon1(2) = gcon1(2)/norm1;

gcon2(1) = gcon2(1)/norm2;
gcon2(2) = gcon2(2)/norm2;

%---
% another way of computing gcon1 and gcon2
% explicitly in terms of gcov1 and gcov2
%---
```

```
gcon1A = ( covmet(2,2)*gcov1 ...
         - covmet(1,2)*gcov2)/covg;

gcon2A = (-covmet(1,2)*gcov1 ...
         + covmet(1,1)*gcov2)/covg;

%---
% another way of computing the covariant base vectors
% explicitly in terms of gcon1 and gcon2
%---

gcov1A = covmet(1,1)*gcon1 + covmet(1,2)*gcon2;
gcov2A = covmet(1,2)*gcon1 + covmet(2,2)*gcon2;

%---
% print
%---

[gcov1  gcov2;
 gcov1A gcov2A;
 gcon1  gcon2;
 gcon1A gcon2A]
```

Running the code generates the following output:

```
    1.3920    0.1493    0.8967    1.5607
    1.3920    0.1493    0.8967    1.5607
    0.7655   -0.4399   -0.0732    0.6828
    0.7655   -0.4399   -0.0732    0.6828
```

as instructed by the last line of the code. As expected, the first pair of lines and the second pair of lines of the output are identical.

Exercises

3.1.1 Prepare a plot of $g \equiv \det(\mathbf{g})$ in the $\xi\eta$ plane corresponding to (3.1.2).

3.1.2 Derive expressions for the covariant base vectors corresponding to the coordinates described in (3.1.2).

3.2 Contravariant base vectors

The contravariant base vectors were given in (3.1.20) in terms of the covariant base vectors, repeated below for convenience,

$$\mathbf{g}^1 \equiv \frac{1}{g}\big(\quad g_{22}\,\mathbf{g}_1 - g_{12}\,\mathbf{g}_2 \big),$$
$$\mathbf{g}^2 \equiv \frac{1}{g}\big(- g_{12}\,\mathbf{g}_1 + g_{11}\,\mathbf{g}_2 \big). \tag{3.2.1}$$

Using expression (3.1.7) for a differential displacement, we find that

$$\mathbf{g}^1 \cdot \mathrm{d}\mathbf{x} = \frac{1}{g}\big(g_{22}\,\mathbf{g}_1 - g_{12}\,\mathbf{g}_2\big) \cdot (\mathbf{g}_1\,\mathrm{d}x^1 + \mathbf{g}_2\,\mathrm{d}x^2). \tag{3.2.2}$$

Carrying out the multiplications, we find that

$$\mathbf{g}^1 \cdot \mathrm{d}\mathbf{x} = \mathrm{d}x^1. \tag{3.2.3}$$

Working in a similar fashion, we find that

$$\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \mathrm{d}x^i \tag{3.2.4}$$

for $i = 1, 2$, as shown in (3.1.21). Integrating this equation between two points, A and B, we find that

$$\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = (x^i)_{\mathrm{B}} - (x^i)_{\mathrm{A}}. \tag{3.2.5}$$

Consequently,

$$\oint \mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = 0, \tag{3.2.6}$$

where the integration is performed along an arbitrary closed path in the xy plane.

3.2.1 Gradient of contravariant coordinates

The contravariant coordinates are functions of position,

$$x^1(\mathbf{x}), \qquad x^2(\mathbf{x}). \tag{3.2.7}$$

Partial derivatives of x^1 and x^2 with respect to Cartesian coordinates, x and y, are well defined. The gradients of these functions are

$$\boldsymbol{\nabla} x^1 = \begin{bmatrix} \partial x^1/\partial x \\ \partial x^1/\partial y \end{bmatrix}, \qquad \boldsymbol{\nabla} x^2 = \begin{bmatrix} \partial x^2/\partial x \\ \partial x^2/\partial y \end{bmatrix}. \tag{3.2.8}$$

By definition,

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1\, \mathrm{d}x^1 + \mathbf{g}_2\, \mathrm{d}x^2 \tag{3.2.9}$$

or

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1\, (\frac{\partial x^1}{\partial x}\, \mathrm{d}x + \frac{\partial x^1}{\partial y}\, \mathrm{d}y) + \mathbf{g}_2\, (\frac{\partial x^2}{\partial x}\, \mathrm{d}x + \frac{\partial x^2}{\partial y}\, \mathrm{d}y), \tag{3.2.10}$$

which is the same as

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1\, \mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla} x^1 + \mathbf{g}_2\, \mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla} x^2. \tag{3.2.11}$$

Setting

$$\mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla} x^1 = \mathrm{d}x^1, \qquad \mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla} x^2 = \mathrm{d}x^2, \tag{3.2.12}$$

and comparing these expressions with those given in (3.1.21), we obtain

$$\mathbf{g}^i = \boldsymbol{\nabla} x^i \tag{3.2.13}$$

for $i = 1, 2$.

Since the curl of the gradient of any function is identically zero, the contravariant base vector fields are irrotational

$$\boldsymbol{\nabla} \times \mathbf{g}^i = \mathbf{0}. \tag{3.2.14}$$

Using the Stokes circulation theorem, we find that the circulation of each contravariant base vector along any arbitrary closed loop in the xy plane is zero, as shown in (3.2.6).

Exercise

3.2.1 Prove that the curl of the gradient of any function is identically zero.

3.3 Covariant coordinates

Consider a one-dimensional field over the x axis, introduce a contravariant coordinate, x^1, specify a function $x(x^1)$, and write

$$\mathbf{g}_1 = g_1\,\mathbf{e}_x, \qquad \mathbf{g}^1 = g^1\,\mathbf{e}_x, \qquad g_1 = \frac{\mathrm{d}x}{\mathrm{d}x^1}, \qquad g^1 = \frac{\mathrm{d}x^1}{\mathrm{d}x}. \tag{3.3.1}$$

Now introduce a covariant coordinate, x_1, and write

$$g^1 = \frac{\mathrm{d}x^1}{\mathrm{d}x} = \alpha\,\frac{\mathrm{d}x}{\mathrm{d}x_1}, \tag{3.3.2}$$

where α is a specified function. Rearranging, we obtain

$$\frac{\mathrm{d}x_1}{\mathrm{d}x} = \alpha\,\frac{\mathrm{d}x}{\mathrm{d}x^1}. \tag{3.3.3}$$

Integrating this equation, we obtain a covariant coordinate distribution, $x_1(x)$, which clearly depends on the specification of α.

3.3.1 Two dimensions

Now we consider curvilinear coordinates in a plane and draw a family of non-intersecting lines that are tangential to the first contravariant base vector, $\mathbf{g}^1$, and another family of non-intersecting lines that are tangential to the second contravariant base vector, $\mathbf{g}^2$.

The position along a line in the first family can be parametrized by a coordinate x_1, and the position along a line in the second family can be parametrized by a coordinate x_2, as shown in Figure 3.1.1.

The doublet (x_1, x_2) comprises *covariant* curvilinear coordinates. In contrast, the doublet (x^1, x^2) associated with the covariant base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ comprises *contravariant* curvilinear coordinates.

3.3.2 Differential displacement

The position vector can be regarded as a function of the covariant coordinates, that is,

$$\mathbf{x}(x_1, x_2). \tag{3.3.4}$$

An infinitesimal displacement in the xy plane can be expressed as

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial \mathbf{x}}{\partial x_2}\,\mathrm{d}x_2, \tag{3.3.5}$$

where $\mathrm{d}x_1$ and $\mathrm{d}x_2$ are differential increments regarded as covariant components of the differential displacement.

Since by definition $\partial\mathbf{x}/\partial x_1$ is parallel to $\mathbf{g}^1$ and $\partial\mathbf{x}/\partial x_2$ is parallel to $\mathbf{g}^2$ at every point, we may write

$$\frac{\partial \mathbf{x}}{\partial x_1} = \frac{1}{\alpha_1}\,\mathbf{g}^1, \qquad \frac{\partial \mathbf{x}}{\partial x_2} = \frac{1}{\alpha_2}\,\mathbf{g}^2, \tag{3.3.6}$$

where $\alpha_1(x_1, x_2)$ and $\alpha_2(x_1, x_2)$ are two appropriate functions. Consequently, the differential displacement is given by

$$\mathrm{d}\mathbf{x} = \frac{1}{\alpha_1}\,\mathbf{g}^1 \mathrm{d}x_1 + \frac{1}{\alpha_2}\,\mathbf{g}^2 \mathrm{d}x_2. \tag{3.3.7}$$

This expression can be contrasted with the corresponding expression involving covariant base vectors and contravariant coordinates, $\mathrm{d}\mathbf{x} = \mathbf{g}_1 \mathrm{d}x^1 + \mathbf{g}_2 \mathrm{d}x^2$.

3.3.3 Relation between covariant and contravariant coordinates

The covariant coordinates, (x_1, x_2), can be regarded as functions of the contravariant coordinates, (x^1, x^2), and *vice versa.* We may write

$$\begin{aligned} \mathrm{d}x_1 &= \frac{\partial x_1}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial x_1}{\partial x^2}\,\mathrm{d}x^2, \\ \mathrm{d}x_2 &= \frac{\partial x_2}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial x_2}{\partial x^2}\,\mathrm{d}x^2. \end{aligned} \tag{3.3.8}$$

Using the chain rule, we write

$$\mathbf{g}_1 \equiv \frac{\partial \mathbf{x}}{\partial x^1} = \frac{\partial \mathbf{x}}{\partial x_1}\frac{\partial x_1}{\partial x^1} + \frac{\partial \mathbf{x}}{\partial x_2}\frac{\partial x_2}{\partial x^1} = \frac{1}{\alpha_1}\,\mathbf{g}^1\frac{\partial x_1}{\partial x^1} + \frac{1}{\alpha_2}\,\mathbf{g}^2\frac{\partial x_2}{\partial x^1} \tag{3.3.9}$$

and

$$\mathbf{g}_2 \equiv \frac{\partial \mathbf{x}}{\partial x^2} = \frac{\partial \mathbf{x}}{\partial x_1}\frac{\partial x_1}{\partial x^2} + \frac{\partial \mathbf{x}}{\partial x_2}\frac{\partial x_2}{\partial x^2} = \frac{1}{\alpha_1}\,\mathbf{g}^1\frac{\partial x_1}{\partial x^2} + \frac{1}{\alpha_2}\,\mathbf{g}^2\frac{\partial x_2}{\partial x^2}. \tag{3.3.10}$$

Projecting these equations onto $\mathbf{g}_1$ or $\mathbf{g}_2$, we find that

$$g_{11} = \frac{1}{\alpha_1}\,\frac{\partial x_1}{\partial x^1}, \qquad g_{12} = \frac{1}{\alpha_1}\,\frac{\partial x_1}{\partial x^2} = \frac{1}{\alpha_2}\,\frac{\partial x_2}{\partial x^1}, \qquad g_{22} = \frac{1}{\alpha_2}\,\frac{\partial x_2}{\partial x^2}, \tag{3.3.11}$$

which imply that

$$g_{11}\,\frac{\partial x_1}{\partial x^2} = g_{12}\,\frac{\partial x_1}{\partial x^1}, \qquad g_{12}\,\frac{\partial x_2}{\partial x^2} = g_{22}\,\frac{\partial x_2}{\partial x^1}. \tag{3.3.12}$$

Substituting (3.3.11) into (3.3.8), we obtain

$$\begin{aligned} \mathrm{d}x_1 &= \alpha_1(g_{11}\,\mathrm{d}x^1 + g_{12}\,\mathrm{d}x^2), \\ \mathrm{d}x_2 &= \alpha_2(g_{12}\,\mathrm{d}x^1 + g_{22}\,\mathrm{d}x^2). \end{aligned} \tag{3.3.13}$$

3.3.4 Compatibility conditions

Equations (3.3.11) require the compatibility conditions

$$\frac{\partial(\alpha_1 g_{11})}{\partial x^2} = \frac{\partial(\alpha_1 g_{12})}{\partial x^1}, \qquad \frac{\partial(\alpha_2 g_{22})}{\partial x^1} = \frac{\partial(\alpha_2 g_{12})}{\partial x^2}. \tag{3.3.14}$$

Expanding the derivatives and rearranging, we find that

$$g_{11}\,\frac{\partial \ln \alpha_1}{\partial x^2} - g_{12}\,\frac{\partial \ln \alpha_1}{\partial x^1} = \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \tag{3.3.15}$$

and

$$g_{22}\,\frac{\partial \ln \alpha_2}{\partial x^1} - g_{12}\,\frac{\partial \ln \alpha_2}{\partial x^2} = \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}. \tag{3.3.16}$$

These compatibility conditions are not necessarily satisfied when $\alpha_1 = 1$ and $\alpha_2 = 1$.

3.3.5 Construction

Once α_1 and α_2 have been constructed in agreement with the compatibility conditions, the covariant coordinates can be deduced from the contravariant coordinates by integrating equations (3.3.11) or (3.3.13).

With reference to the grid in the $x^1 x^2$ plane shown in Figure 3.1.2(a), we may specify arbitrarily the distribution of α_1 along the bottom grid line, $x^2 = 0$, and the distribution of α_2 along the left grid line, $x^1 = 0$. In the second step, we may apply forward difference approximations to the partial derivatives in (3.3.15) to obtain the distribution of α_1 along the second from the bottom grid line.

Similarly, we may apply forward difference approximations to the partial derivatives in (3.3.16) to obtain the distribution of α_2 along the second from the bottom grid line. The procedure may then be repeated to construct the entire covariant coordinate nodal field.

3.3.6 Orthogonal coordinates

The compatibility conditions for orthogonal coordinates require that

$$\frac{\partial(\alpha_1\, g_{11})}{\partial x^2} = 0, \qquad \frac{\partial(\alpha_2\, g_{22})}{\partial x^1} = 0. \tag{3.3.17}$$

Integrating these equations, we obtain

$$\alpha_1 = \frac{1}{g_{11}}\,\mathcal{A}(x^1), \qquad \alpha_2 = \frac{1}{g_{22}}\,\mathcal{B}(x^2), \tag{3.3.18}$$

where $\mathcal{A}(x^1)$ and $\mathcal{B}(x^2)$ are two arbitrary functions. Integrating equations (3.3.11), we obtain

$$x_1 = \int \mathcal{A}(x^1)\,\mathrm{d}x^1, \qquad x_2 = \int \mathcal{B}(x^2)\,\mathrm{d}x^2. \tag{3.3.19}$$

For $\mathcal{A}(x^1) = 1$ and $\mathcal{B}(x^2) = 1$, we find that $x_1 = x^1$, $x_2 = x^2$, and

$$\frac{\partial \mathbf{x}}{\partial x_1} = g_{11}\,\mathbf{g}^1, \qquad \frac{\partial \mathbf{x}}{\partial x_2} = g_{22}\,\mathbf{g}^2. \tag{3.3.20}$$

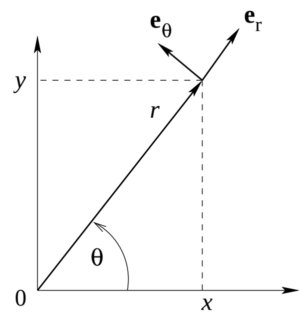

FIGURE 3.3.1 Illustration of plane polar coordinates, (r, θ), in the xy plane defined with respect to Cartesian coordinates, (x, y).

3.3.7 *Plane polar coordinates*

In the case of plane polar coordinates, $x^1 = r$ and $x^2 = \theta$, where r is the distance from the original and θ is the polar angle measured around the origin, as shown in Figure 3.3.1. The Cartesian coordinates are given by $x = r\cos\theta$ and $x = r\sin\theta$.

The covariant and contravariant base vectors are given by

$$\mathbf{g}_r = \mathbf{e}_r, \qquad \mathbf{g}_\theta = r\,\mathbf{e}_\theta, \qquad \mathbf{g}^r = \mathbf{e}_r, \qquad \mathbf{g}^\theta = \frac{1}{r}\,\mathbf{e}_\theta, \quad (3.3.21)$$

and the covariant metric coefficients are given by

$$g_{rr} = 1, \qquad g_{r\theta} = 0, \qquad g_{\theta\theta} = r^2. \qquad (3.3.22)$$

The compatibility conditions (3.3.17) require that

$$\frac{\partial \alpha_1}{\partial \theta} = 0, \qquad \frac{\partial(\alpha_2 r^2)}{\partial r} = 0, \qquad (3.3.23)$$

and therefore

$$\alpha_1 = \mathcal{A}(r), \qquad \alpha_2 = \frac{1}{r^2}\,\mathcal{B}(\theta), \qquad (3.3.24)$$

where $\mathcal{A}(r)$ and $\mathcal{B}(\theta)$ are arbitrary functions. Integrating equations (3.3.11), we obtain

$$x_1 = \int \mathcal{A}(r)\,\mathrm{d}r, \qquad x_2 = \int \mathcal{B}(\theta)\,\mathrm{d}\theta. \qquad (3.3.25)$$

Equations (3.3.6) become

$$\frac{\partial \mathbf{x}}{\partial x_1} = \frac{1}{\mathcal{A}(r)}\,\mathbf{e}_r, \qquad \frac{\partial \mathbf{x}}{\partial x_2} = \frac{1}{\mathcal{B}(\theta)}\,r\,\mathbf{e}_\theta. \tag{3.3.26}$$

We may set $\mathcal{A}(r) = 1$ and $\mathcal{B}(\theta) = 1$ to obtain $x_1 = r$ and $x_2 = \theta$. These results emphasize that, for given contravariant coordinates, the associated covariant coordinates are not uniquely defined.

Exercise

3.3.1 Derive the expressions shown in (3.3.21).

3.4 Metric coefficients

The covariant components of the metric tensor, also called the covariant metric coefficients, were defined in Section 3.1.5 in terms of the covariant base vectors as

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j. \tag{3.4.1}$$

Explicitly,

$$g_{11} = \mathbf{g}_1 \cdot \mathbf{g}_1, \qquad g_{12} = g_{21} = \mathbf{g}_1 \cdot \mathbf{g}_2, \qquad g_{22} = \mathbf{g}_2 \cdot \mathbf{g}_2. \tag{3.4.2}$$

It is useful to collect these coefficients into a symmetric matrix, as shown in (3.1.9),

$$\mathbf{g} \equiv \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \tag{3.4.3}$$

The contravariant components of the metric tensor, also called the contravariant metric coefficients, are defined in a similar fashion in terms of the contravariant base vectors as

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j. \tag{3.4.4}$$

Explicitly,

$$g^{11} = \mathbf{g}^1 \cdot \mathbf{g}^1, \qquad g^{12} = g^{21} = \mathbf{g}^1 \cdot \mathbf{g}^2, \qquad g^{22} = \mathbf{g}^2 \cdot \mathbf{g}^2. \tag{3.4.5}$$

It is useful to collect these coefficients into a symmetric matrix,

$$\boldsymbol{\gamma} \equiv \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}, \tag{3.4.6}$$

so that $\gamma_{ij} = b^{ij}$. In fact, we will show that the matrix of contravariant metric coefficients is the inverse of that of the covariant metric coefficients.

3.4.1 Orthogonality of metric coefficients matrices

Using the first of expressions (3.1.20) for $\mathbf{g}^1$, we find that

$$g^{11} \equiv \mathbf{g}^1 \cdot \mathbf{g}^1 = \frac{1}{g^2} \left(g_{22}\, \mathbf{g}_1 - g_{12}\, \mathbf{g}_2 \right) \cdot \left(g_{22}\, \mathbf{g}_1 - g_{12}\, \mathbf{g}_2 \right). \tag{3.4.7}$$

Carrying out the multiplications and simplifying, we find

$$g^{11} = \frac{g_{22}}{g}. \tag{3.4.8}$$

Working in a similar fashion with the first and second expressions in (3.1.20), we find that

$$g^{11} = \frac{g_{22}}{g}. \qquad g^{22} = \frac{g_{11}}{g}, \qquad g^{12} = -\frac{g_{12}}{g}. \tag{3.4.9}$$

These expressions reveal that the matrix $\mathbf{g}$ is the inverse of matrix $\boldsymbol{\gamma}$, and *versa versa*.

$$\mathbf{g} = \boldsymbol{\gamma}^{-1}, \qquad \boldsymbol{\gamma} = \mathbf{g}^{-1}, \tag{3.4.10}$$

where the superscript -1 denotes the matrix inverse.

Referring to (3.1.20), we find that

$$\mathbf{g}^1 = g^{11}\, \mathbf{g}_1 + g^{12}\, \mathbf{g}_2, \qquad \mathbf{g}^2 = g^{12}\, \mathbf{g}^1 + g^{22}\, \mathbf{g}_2, \tag{3.4.11}$$

which provide us with a formula for raising indices,

$$\mathbf{g}^i = g^{ij} \mathbf{g}_j. \tag{3.4.12}$$

In Section 3.1, we found a corresponding formula for lowering indices, $\mathbf{g}_i = g_{ij}\mathbf{g}^j$.

3.4.2 Confirmation by code

The following Matlab code, continuing the code *nonortho* listed in Section 3.1, located in directory NONORTHO of TUNLIB, computes the contravariant components of the metric tensor and confirms the aforementioned orthogonality:

```
conmet(1,1) = gcon1(1)*gcon1(1) + gcon1(2)*gcon1(2);
conmet(1,2) = gcon1(1)*gcon2(1) + gcon1(2)*gcon2(2);
conmet(2,1) = conmetric(1,2);
conmet(2,2) = gcon2(1)*gcon2(1) + gcon2(2)*gcon2(2);

[inv(covmet) conmet]
```

Running the code generates the following output:

```
   0.7795   -0.3564    0.7795   -0.3564
  -0.3564    0.4716   -0.3564    0.4716
```

as instructed by the last line of the code. As expected, the first two columns are the same as the last two columns.

3.4.3 Mixed components of the metric tensor

For completeness, we introduce the mixed components of the metric tensor defined as

$$g_i^{\circ j} \equiv \mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{g}_j \cdot \mathbf{g}^i = g_j^{\circ i} = \delta_{ij}, \tag{3.4.13}$$

stemming from the biorthogonality condition; we recall that $\circ$ is a blank space holder. The mixed components can be collected into two matrices, $[g_i^{\circ j}]$ and $[g_j^{\circ i}]$, which are both equal to the identity tensor.

To conform with rules for raising and lowering indices, equations (3.4.13) are typically written as

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^{\circ j}, \qquad \mathbf{g}^i \cdot \mathbf{g}_j = \delta_{\circ j}^{i}, \tag{3.4.14}$$

where $\delta_i^{\circ j}$ and $\delta_{\circ j}^{i}$ are disguised Kronecker deltas.

3.4.4 The metric tensor is the identity tensor

The metric tensor may now be defined in the usual way in terms of its contravariant, covariant, or mixed components,

$$\mathbf{G} = g^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j = g^{i}_{\circ j}\,\mathbf{g}_i \otimes \mathbf{g}^j = g_i^{\circ j}\,\mathbf{g}^i \otimes \mathbf{g}_j = g_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j, \tag{3.4.15}$$

where g^{ij} are contravariant components, g_{ij} are covariant components, and summation is implied over the repeated indices, i and j. We find that

$$\mathbf{G}\cdot\mathbf{g}^m = g^{ij}\,\mathbf{g}_i\,\delta_{jm} = g^{i}_{\circ j}\,g^{jm} = g_i^{\circ j}\,\mathbf{g}^i\,\delta_{jm} = g_{ij}\,\mathbf{g}^i\,g^{jm} = \mathbf{g}^m, \tag{3.4.16}$$

for any contravariant base vector, $\mathbf{g}^m$ and also $\mathbf{G}\cdot\mathbf{g}_m = \mathbf{g}_m$, for any covariant base vector, $\mathbf{g}_m$, which confirm that the metric tensor is the identity tensor, $\mathbf{G} = \mathbf{I}$.

The following Matlab code, continuing the code *nonortho* listed previously in this section, located in directory NONORTHO of TUNLIB, confirms that the metric tensor is the identity tensor:

```
for m=1:2
 for n=1:2
  gmet1(m,n) =  ...
   conmet(1,1) * gcov1(m)*gcov1(n) ...
  +conmet(1,2) * gcov1(m)*gcov2(n) ...
  +conmet(2,1) * gcov2(m)*gcov1(n) ...
  +conmet(2,2) * gcov2(m)*gcov2(n);

  gmet2(m,n) =  ...
   covmet(1,1) * gcon1(m)*gcon1(n) ...
  +covmet(1,2) * gcon1(m)*gcon2(n) ...
  +covmet(2,1) * gcon2(m)*gcon1(n) ...
  +covmet(2,2) * gcon2(m)*gcon2(n);

  gmet3(m,n) =  ...
   gcon1(m)*gcov1(n) ...
  +gcon2(m)*gcov2(n);
```

```
 end
end

[gmet1 gmet2 gmet3]
```

Running the code generates the following output:

```
   1.0000    0.0000    1.0000    0.0000    1.0000    0.0000
   0.0000    1.0000    0.0000    1.0000    0.0000    1.0000
```

as instructed by the last line of the code. Each of the three pairs of columns is the identity matrix.

3.4.5 Summary of notation and definitions

Notation and miscellaneous definitions are listed in Table 3.4.1 for ready reference.

3.4.6 Traces

Because of the biorthonormality of the covariant and contravariant base vectors, the trace of the tensors $\mathbf{g}_i \otimes \mathbf{g}^j$ and $\mathbf{g}^j \otimes \mathbf{g}_j$ satisfy

$$\text{trace}(\mathbf{g}_i \otimes \mathbf{g}^j) = \delta_{ij}, \qquad \text{trace}(\mathbf{g}^i \otimes \mathbf{g}_j) = \delta_{ij}, \tag{3.4.17}$$

where δ_{ij} is Kronecker's delta. By definition, the traces of the matrices $\mathbf{g}_i \otimes \mathbf{g}_j$ and $\mathbf{g}^i \otimes \mathbf{g}^j$ are

$$\begin{aligned} \text{trace}(\mathbf{g}_i \otimes \mathbf{g}_j) &= \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}, \\ \text{trace}(\mathbf{g}^i \otimes \mathbf{g}^j) &= \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij}. \end{aligned} \tag{3.4.18}$$

The right-hand sides are the covariant or contravariant components of the metric tensor (identity matrix.)

Exercise

3.4.1 Confirm that the orthogonality property (3.4.10) is consistent with expressions (3.4.9).

$\mathbf{g}_1, \mathbf{g}_2$	covariant base vectors
x^1, x^2	contravariant coordinates
$\mathbf{g}^1, \mathbf{g}^2$	contravariant base vectors
x_1, x_2	covariant coordinates
$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$	covariant metric tensor components
$\mathbf{e}_1 = \mathbf{g}_1/\sqrt{g_{11}}$, $\mathbf{e}_2 = \mathbf{g}_2/\sqrt{g_{22}}$	covariant unit base vectors
$\mathbf{g} = [g_{ij}]$	covariant metric coefficients matrix
$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$	contravariant metric tensor components
$\mathbf{e}^1 = \mathbf{g}^1/\sqrt{g^{11}}$, $\mathbf{e}^2 = \mathbf{g}^2/\sqrt{g^{22}}$	contravariant base unit vectors
$\boldsymbol{\gamma} = [g^{ij}]$	contravariant metric coefficients matrix

TABLE 3.4.1 Definitions of covariant and contravariant coordinates, base vectors, and vector components in two dimensions. Similar definitions are made in three dimensions, as discussed in Chapter 4.

3.5 Areal and coordinate-line metrics

Consider a small shaded surface element in the xy plane defined by two adjacent pairs of contravariant coordinate lines, as shown in Figure 3.5.1. For convenience, we set

$$a \equiv (\mathbf{g}_1)_x, \qquad b \equiv (\mathbf{g}_1)_y, \qquad c \equiv (\mathbf{g}_2)_x, \qquad d \equiv (\mathbf{g}_2)_y, \tag{3.5.1}$$

where $(\mathbf{g}_i)_\alpha$ is the α component of $\mathbf{g}_i$ for $i = 1, 2$ and $\alpha = x, y$. The area of the shaded element is

$$\mathrm{d}A = |\mathbf{g}_1 \mathrm{d}x^1 \times \mathbf{g}_2 \mathrm{d}x^2| = \mathcal{J}\, \mathrm{d}x^1\, \mathrm{d}x^2, \tag{3.5.2}$$

where $\mathcal{J} = ad - bc$ is the areal metric.

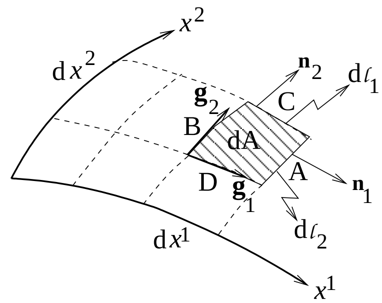

FIGURE 3.5.1 Illustration of a differential element defined by two pairs of contravariant coordinate lines, showing the arc lengths of two edges and associated unit normal vectors.

The covariant matrix coefficients are given by $g_{11} = a^2 + b^2$, $g_{12} = ac + bd$, and $g_{22} = c^2 + d^2$. We may confirm by direct substitution that

$$\mathcal{J} = \sqrt{g}, \tag{3.5.3}$$

where

$$g \equiv \det(\mathbf{g}) = g_{11}g_{22} - g_{12}^2 = (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2. \tag{3.5.4}$$

Carrying out the multiplications on the right-hand side, we find the expression $(ad - bc)^2 = \mathcal{J}^2$.

3.5.1 Coordinate line metrics

The differential arc lengths $\mathrm{d}\ell_1$ and $\mathrm{d}\ell_2$ shown in Figure 3.5.1 are given by

$$\mathrm{d}\ell_1 = \sqrt{g_{11}}\,\mathrm{d}x^1, \qquad \mathrm{d}\ell_2 = \sqrt{g_{22}}\,\mathrm{d}x^2. \tag{3.5.5}$$

The associated unit normal vectors shown in Figure 3.5.1 are parallel to the contravariant base vectors, given by

$$\mathbf{n}_1 = \frac{1}{\sqrt{g^{11}}}\,\mathbf{g}^1, \qquad \mathbf{n}_2 = \frac{1}{\sqrt{g^{22}}}\,\mathbf{g}^2. \tag{3.5.6}$$

Note that, in contrast to base vectors, the unit normal vectors are dimensionless.

3.5.2 Coordinate flux of a vector field

Using the preceding expressions, we find that, if $\mathbf{v}$ is an arbitrary vector field, then

$$\mathbf{v} \cdot \mathbf{n}_1 \, \mathrm{d}\ell_2 = \mathbf{v} \cdot \mathbf{g}^1 \sqrt{\frac{g_{22}}{g^{11}}} \, \mathrm{d}x^2. \tag{3.5.7}$$

Now recalling that $g^{11} = g_{22}/g$, as shown in (3.4.9), we obtain

$$\mathbf{v} \cdot \mathbf{n}_1 \, \mathrm{d}\ell_2 = \mathbf{v} \cdot \mathbf{g}^1 \sqrt{g} \, \mathrm{d}x^2. \tag{3.5.8}$$

If v_1 is the first contravariant component of $\mathbf{v}$ defined by

$$\mathbf{v} = v^1 \, \mathbf{g}_1 + v^2 \, \mathbf{g}_2, \tag{3.5.9}$$

then $\mathbf{v} \cdot \mathbf{g}^1 = v^1$ and thus

$$\mathbf{v} \cdot \mathbf{n}_1 \, \mathrm{d}\ell_1 = v^1 \sqrt{\frac{g_{22}}{g^{11}}} \, \mathrm{d}x^2, \tag{3.5.10}$$

yielding

$$\mathbf{v} \cdot \mathbf{n}_1 \, \mathrm{d}\ell_2 = v^1 \sqrt{g} \, \mathrm{d}x^2. \tag{3.5.11}$$

Note that the areal metric, $\sqrt{g}$, arises naturally in this expression.

Working in a similar fashion, we find that

$$\mathbf{v} \cdot \mathbf{n}_2 \, \mathrm{d}\ell_1 = \mathbf{v} \cdot \mathbf{g}^2 \sqrt{\frac{g_{11}}{g^{22}}} \, \mathrm{d}x^1, \tag{3.5.12}$$

and then

$$\mathbf{v} \cdot \mathbf{n}_2 \, \mathrm{d}\ell_2 = v^2 \sqrt{g} \, \mathrm{d}x^1, \tag{3.5.13}$$

which is the counterpart of (3.5.11).

Expressions (3.5.11) and (3.5.13) can be used to compute the integral of the normal component, $\mathbf{v} \cdot \mathbf{n}$, around the four edges of the shaded area in Figure 3.5.1. Physically, the integral represents an integrated convective flux associated with a velocity field, $\mathbf{v}$.

3.5.3 Divergence of a vector field

Applying the divergence theorem for an arbitrary vector field, $\mathbf{v}$, over the shaded area shown in Figure 3.5.1, we write

$$\iint \boldsymbol{\nabla} \cdot \mathbf{v} \, \mathrm{d}A = \oint \mathbf{n} \cdot \mathbf{v} \, \mathrm{d}\ell, \tag{3.5.14}$$

where the line integral on the right-hand side is computed along the four edges of the shaded area.

$$\iint \boldsymbol{\nabla} \cdot \mathbf{v} \, \mathcal{J} \, \mathrm{d}x^1 \, \mathrm{d}x^2 = \int_{\mathrm{A,B,C,D}} \mathbf{n} \cdot \mathbf{v} \, \mathrm{d}\ell, \tag{3.5.15}$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ denote the left, right, top, and bottom edge of the shaded area.

Approximating the line integrals along the edges for a small shaded shell with the expressions provided in (3.5.11) and (3.5.13), we obtain

$$\begin{aligned}(\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathcal{J} \, \mathrm{d}x^1 \, \mathrm{d}x^2 = {} & (v^1 \sqrt{g})_{x^1+\mathrm{d}x^1} \, \mathrm{d}x^2 - (v^1 \sqrt{g})_{x^1} \, \mathrm{d}x^2 \\ & + (v^2 \sqrt{g})_{x^2+\mathrm{d}x^2} \, \mathrm{d}x^1 - (v^2 \sqrt{g})_{x^2} \, \mathrm{d}x^1, \end{aligned} \tag{3.5.16}$$

where the divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$ on the left-hand side is evaluated at a designated center of the shaded area. Now setting $\mathcal{J} = \sqrt{g}$, dividing both sides by the product $\mathrm{d}x^1 \, \mathrm{d}x^2$, and taking the limit as $\mathrm{d}x^1$ and $\mathrm{d}x^2$ tend to zero, we obtain an expression for the divergence of a vector field,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{\sqrt{g}} \Big(\frac{\partial}{\partial x^1} (v^1 \sqrt{g}) + \frac{\partial}{\partial x^2} (v^2 \sqrt{g}) \Big). \tag{3.5.17}$$

In compact notation,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (v^i \sqrt{g}), \tag{3.5.18}$$

where summation is implied over the repeated index, i. The derivatives on the right-hand side can be discretized by standard numerical methods on a curvilinear grid.

3.5.4 Finite-volume method

As an application, we consider the continuity equation for a general compressible or incompressible fluid,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = 0, \tag{3.5.19}$$

where ρ is the fluid density, $\mathbf{u}$ is the fluid velocity, and t stands for time.

Integrating this equation over the shaded area shown in Figure 3.5.1, and applying the divergence theorem stated in (3.5.14) for $\mathbf{v} = \rho\mathbf{u}$, we obtain

$$\iint \frac{\partial \rho}{\partial t}\, \mathrm{d}A + \oint \rho\, \mathbf{n} \cdot \mathbf{u}\, \mathrm{d}\ell = 0, \tag{3.5.20}$$

where the line integral is computed along the four edges of the shaded area. Using the formulas derived previously in this section, we obtain

$$\begin{aligned} \oint \rho\, \mathbf{n} \cdot \mathbf{u}\, \mathrm{d}\ell = & \int_{\mathrm{A}} \rho\, u^1 \sqrt{g}\, \mathrm{d}x^2 - \int_{\mathrm{B}} \rho\, u^1 \sqrt{g}\, \mathrm{d}x^2 \\ & + \int_{\mathrm{C}} \rho\, u^2 \sqrt{g}\, \mathrm{d}x^1 - \int_{\mathrm{D}} \rho\, u^2 \sqrt{g}\, \mathrm{d}x^1, \end{aligned} \tag{3.5.21}$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ denote the left, right, top, and bottom edge of the shaded area.

In a finite-volume formulation, edge distributions are replaced by a representative constant value defined either in terms of neighboring shaded areas (cells or finite volumes) or in terms of neighboring nodes. Approximating the areal integral in (3.5.20) over a cell with the expression

$$\iint \frac{\partial \rho}{\partial t}\, \mathrm{d}A \simeq \big(\frac{\partial \rho}{\partial t} \sqrt{g} \big)_{\mathrm{cell}}\, \Delta x^1\, \Delta x^2, \tag{3.5.22}$$

we obtain

$$\begin{aligned} \big(\frac{\partial \rho}{\partial t} \sqrt{g}\big)_{\mathrm{cell}}\, \Delta x^1\, \Delta x^2 + & \Big((\rho\, u^1 \sqrt{g})_{\mathrm{A}} - (\rho\, u^1 \sqrt{g})_{\mathrm{B}} \Big)\, \Delta x^2 \\ & + \Big((\rho\, u^2 \sqrt{g})_{\mathrm{C}} - (\rho\, u^2 \sqrt{g})_{\mathrm{D}} \Big)\, \Delta x^1 = 0. \end{aligned} \tag{3.5.23}$$

However, since curvilinear coordinates provide us with a structured grid that can be used to implement finite-difference discretizations, a compelling argument for using the finite-volume method in curvilinear coordinates is hard to make.

3.5.5 Laplacian of a scalar field

The Laplacian of a scalar field, f, is a scalar field defined as the divergence of the gradient of the field,

$$\nabla^2 f \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f). \tag{3.5.24}$$

Applying (3.5.18) for $\mathbf{v} = \boldsymbol{\nabla} f$, we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \big(\sqrt{g}\, (\boldsymbol{\nabla} f)^i \big), \tag{3.5.25}$$

where $(\boldsymbol{\nabla} f)^i$ is the ith contravariant component of the gradient. In Section 5.1, we will show that

$$(\boldsymbol{\nabla} f)^i = \frac{\partial f}{\partial x_i} = g^{ki} \frac{\partial f}{\partial x^k}, \tag{3.5.26}$$

as shown in (5.1.8), where summation is implied over the repeated index, k. Consequently, the Laplacian is given by

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \Big(\sqrt{g}\, g^{ki} \frac{\partial f}{\partial x^k} \Big), \tag{3.5.27}$$

where summation is implied over the repeated indices, i and k. Note that the contravariant metric coefficients are involved in this expression.

Expanding the derivatives on the right-hand side of (3.5.27) and rearranging, we obtain

$$\nabla^2 f = g^{ki} \frac{\partial^2 f}{\partial x^i \partial x^k} + v^k \frac{\partial f}{\partial x^k}, \tag{3.5.28}$$

where

$$v^k \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \Big(g^{ki} \sqrt{g} \Big) \tag{3.5.29}$$

and summation is implied over the repeated index, k. The first and second derivatives on the right-hand side of (3.5.28) can be discretized by standard numerical methods. Explicitly, the Laplacian is given by

$$\nabla^2 f = g^{\xi\xi} \frac{\partial^2 f}{\partial \xi^2} + 2\, g^{\xi\eta} \frac{\partial^2 f}{\partial \xi \partial \eta} + g^{\eta\eta} \frac{\partial^2 f}{\partial \eta^2} + v^{\xi} \frac{\partial f}{\partial \xi} + v^{\eta} \frac{\partial f}{\partial \eta}, \quad (3.5.30)$$

where ξ stands for x^1 and η stands for x^2. First and second derivatives are involved in this expression.

Exercise

3.5.1 Confirm that the Laplacian given in (3.5.27) provides us with the sum of the second derivatives in Cartesian coordinates.

3.6 Oblique rectilinear coordinates

The simplest non-Cartesian coordinates are oblique rectilinear coordinates in the xy plane defined by inclined rectilinear coordinate lines, as shown in Figure 3.6.1. The covariant base consist of two dimensionless constant vectors,

$$\mathbf{g}_1 = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \qquad \mathbf{g}_2 = \begin{bmatrix} \beta \\ 1 \end{bmatrix}, \quad (3.6.1)$$

where α ad β are arbitrary positive or negative dimensionless constants and the square brackets enclose the x and y Cartesian coordinates. Oblique rectilinear coordinates are sometimes called *nonorthogonal homogeneous coordinates.*

The parameters α and β are related to the angles ψ and ϕ shown in Figure 3.6.1 by

$$\alpha = \tan\psi, \qquad \beta = \tan\phi. \quad (3.6.2)$$

When $\alpha + \beta = 0$, and correspondingly $\phi = -\psi$, we obtain rotated Cartesian coordinates.

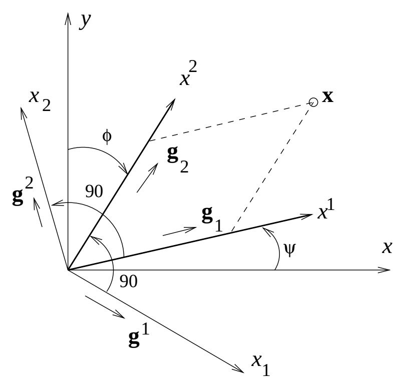

FIGURE 3.6.1 Illustration of two-dimensional oblique rectilinear coordinates, x^1 and x^2 corresponding to the covariant base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$.

3.6.1 Relation between coordinates

Integrating equation

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1 \,\mathrm{d}x^1 + \mathbf{g}_2 \,\mathrm{d}x^2, \tag{3.6.3}$$

we find that

$$x = x^1 + \beta x^2, \qquad y = \alpha x^1 + x^2. \tag{3.6.4}$$

Conversely, using Cramer's rule to solve these equations for x^1 and x^2, we find that

$$x^1 = \frac{1}{1-\alpha\beta}\,(x - \beta y), \qquad x^2 = \frac{1}{1-\alpha\beta}\,(y - \alpha x). \tag{3.6.5}$$

Note that $x_2 = 0$ when $y = \alpha x$ and $x_1 = 0$ when $x = \beta y$.

3.6.2 Covariant metric coefficients

The covariant components of the metric tensor can be arranged in the following matrix:

$$\mathbf{g} = \begin{bmatrix} 1+\alpha^2 & \alpha+\beta \\ \alpha+\beta & 1+\beta^2 \end{bmatrix}, \tag{3.6.6}$$

The determinant of this matrix is

$$g = \det(\mathbf{g}) = (1 - \alpha\beta)^2. \tag{3.6.7}$$

The areal metric is

$$\mathcal{J} = \sqrt{g} = |1 - \alpha\beta|, \tag{3.6.8}$$

where the vertical bar denote the absolute value. A singularity arises only when $\alpha\beta = 1$, whereupon $\mathbf{g}_1 = \mathbf{g}_2$.

3.6.3 Contravariant base vectors

Referring to formulas (3.1.20), we find that the contravariant base vectors are given by

$$\mathbf{g}^1 = \frac{1}{(1-\alpha\beta)^2} \big((1+\beta^2)\, \mathbf{g}_1 - (\alpha+\beta)\, \mathbf{g}_2 \big) \tag{3.6.9}$$

and

$$\mathbf{g}^2 = \frac{1}{(1-\alpha\beta)^2} \big(-(\alpha+\beta)\, \mathbf{g}_1 + (1+\alpha^2)\, \mathbf{g}_2 \big). \tag{3.6.10}$$

Making substitutions, we find that

$$\mathbf{g}^1 = \frac{1}{1-\alpha\beta} \begin{bmatrix} 1 \\ -\beta \end{bmatrix}, \qquad \mathbf{g}^2 = \frac{1}{1-\alpha\beta} \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}. \tag{3.6.11}$$

A singularity arises only when $\alpha\beta = 1$, whereupon $\mathbf{g}_1 = \mathbf{g}_2$. These expressions confirm that $\mathbf{g}^1 = \boldsymbol{\nabla}\mathbf{x}^1$ and $\mathbf{g}^2 = \boldsymbol{\nabla}\mathbf{x}^2$, where the function $\mathbf{v}(x^1, x^2)$ is described in (3.6.5).

3.6.4 Contravariant metric coefficients

The contravariant components of the metric tensor can be arranged in the following matrix:

$$\boldsymbol{\gamma} = \frac{1}{(1-\alpha\beta)^2} \begin{bmatrix} 1+\beta^2 & -\alpha-\beta \\ -\alpha-\beta & 1+\alpha^2 \end{bmatrix}. \tag{3.6.12}$$

It can be confirmed by direct substitution that the matrix $\boldsymbol{\gamma}$ is the inverse of the matrix g shown in (3.6.6), and *vice versa*.

3.6.5 Laplacian of a scalar field

Referring to formula (3.5.30), we find that the Laplacian of a scalar field, $f(\mathbf{x})$, is given by the formula

$$\nabla^2 f = \frac{1}{(1-\alpha\beta)^2}\Big((1+\beta^2)\,\frac{\partial^2 f}{\partial x^1 \partial x^1} \\ -2\,(\alpha+\beta)\,\frac{\partial^2 f}{\partial x^1 \partial x^2} + (1+\alpha^2)\,\frac{\partial^2 f}{\partial x^2 \partial x^2}\,\Big). \qquad (3.6.13)$$

The presence of a mixed second derivatives is a feature of non-orthogonal coordinates.

3.6.6 Laplace's equation

Consider Laplace's equation,

$$\nabla^2 f = 0, \qquad (3.6.14)$$

subject to specified boundary conditions. Invoking the expression for the Laplacian given in (3.6.13), we obtain

$$(1+\beta^2)\,\frac{\partial^2 f}{\partial x^1 \partial x^1} - 2\,(\alpha+\beta)\,\frac{\partial^2 f}{\partial x^1 \partial x^2} + (1+\alpha^2)\,\frac{\partial^2 f}{\partial x^2 \partial x^2} = 0. \qquad (3.6.15)$$

Two simple solutions are $f = cx^1$ and $f = cx^2$, where c is a constant. The quadratic function $f = cx^1x^2$ is *not* a solution.

3.6.7 Separation of variables

We may attempt to solve Laplace's equation by separation of variables in x^1 and x^2, setting

$$f(x^1, x^2) = \Phi(x^1)\,\Psi(x^2). \qquad (3.6.16)$$

Making substitutions, we find that

$$(1+\beta^2)\,\frac{\Phi''}{\Phi} - 2\,(\alpha+\beta)\,\frac{\Phi'\Psi'}{\Phi\Psi} + (1+\alpha^2)\,\frac{\Psi''}{\Psi} = 0. \qquad (3.6.17)$$

The first term on the left-hand side is a function of x^1, the last term is a function of x^2, and the second term is a function of x^1 and x^2. We conclude that Laplace's equation is non-separable in oblique rectilinear coordinates.

3.6.8 Small obliqueness

We may define

$$\varepsilon \equiv \tfrac{1}{2}\,(\alpha+\beta), \qquad \gamma \equiv \tfrac{1}{2}\,(\beta-\alpha), \tag{3.6.18}$$

and write

$$\alpha = \varepsilon - \gamma, \qquad \beta = \varepsilon + \gamma. \tag{3.6.19}$$

Laplace's equation becomes

$$(1+\gamma^2+2\,\varepsilon\,\gamma+\varepsilon^2)\,\frac{\partial^2 f}{\partial x^1\partial x^1} - 4\varepsilon\,\frac{\partial^2 f}{\partial x^1\partial x^2} + (1+\gamma^2-2\,\varepsilon\,\gamma+\varepsilon^2)\,\frac{\partial^2 f}{\partial x^2\partial x^2} = 0. \tag{3.6.20}$$

Rearranging, we obtain

$$\widehat{\nabla}^2 f = \varepsilon\,\frac{2}{1+\gamma^2+\varepsilon^2}\big(-\gamma\,\frac{\partial^2 f}{\partial x^1\partial x^1} + 2\,\frac{\partial^2 f}{\partial x^1\partial x^2} + \gamma\,\frac{\partial^2 f}{\partial x^2\partial x^2}\,\big), \tag{3.6.21}$$

where

$$\widehat{\nabla}^2 \equiv \frac{\partial^2 f}{\partial x^1\partial x^1} + \frac{\partial^2 f}{\partial x^2\partial x^2} \tag{3.6.22}$$

is the Laplacian in the curvilinear coordinate plane. We recall that, when $\varepsilon = 0$, we obtain rotated orthogonal coordinates.

3.6.9 Perturbation expansion

A perturbation expansion can be written for small ϵ, setting $f = f_0 + \epsilon\, f_1 + \cdots$. Substituting this expansion into (3.6.21), we obtain

$$\widehat{\nabla}^2 f_0 = 0 \tag{3.6.23}$$

and

$$\widehat{\nabla}^2 f_1 = \frac{2}{1+\gamma^2}\,\big(-\gamma\frac{\partial^2 f_0}{\partial x^1\partial x^1} + 2\,\frac{\partial^2 f_0}{\partial x^1\partial x^2} + \gamma\,\frac{\partial^2 f_0}{\partial x^2\partial x^2}\,\big). \tag{3.6.24}$$

Similar equations can be derived for higher-order corrections.

3.6.10 Example

As an example, we consider the solution for boundary conditions computed from $f = \xi x^1 x^2$ for $x^1 = \pm a$ and $x^2 = \pm b$, where ξ, a, and b are three constants. The solution of (3.6.23) subject to these boundary conditions is found readily by inspection, and is given by

$$f_0 = \xi x^1 x^2. \tag{3.6.25}$$

Equation (3.6.24) becomes

$$\frac{\partial^2 f_1}{\partial x^1 \partial x^1} + \frac{\partial^2 f_1}{\partial x^2 \partial x^2} = \frac{4\xi}{1+\gamma^2}, \tag{3.6.26}$$

which is a Poisson equation with a constant forcing term on the right-hand side. The boundary conditions are $f_1 = 0$ at $x^1 = \pm a$ and $x^2 = \pm b$.

The solution can be found by Fourier expansions, and is given by

$$f_1(\xi, \eta) = \frac{2\xi}{1+\gamma^2}\, b^2\, \Psi(\xi, \eta), \tag{3.6.27}$$

where $\xi = x^1$, $\eta = x^2$,

$$\Psi(\xi, \eta) = 1 - \frac{\eta^2}{b^2} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{1}{\alpha_n^3} \frac{\cosh(\alpha_n\, \xi/b)}{\cosh(\alpha_n\, a/b)} \cos(\alpha_n \frac{\eta}{b}), \tag{3.6.28}$$

and $\alpha_n = (n - \frac{1}{2})\,\pi$.

Exercises

3.6.1 Confirm that $[g_{ij}]$ is the inverse of $[g^{ij}]$.

3.6.2 Write an equation for the second-order solution f_2 in the asymptotic expansion for ε for Laplace's equation.

3.7 Canonical oblique rectilinear coordinates

In the canonical state of oblique coordinates, $\alpha = 0$, the parameter β is arbitrary, and $\varepsilon = \frac{1}{2}\beta$, as shown in Figure 3.7.1. The Cartesian coordinates are related to the canonical oblique coordinates by

$$x = x^1 + \beta x^2, \qquad y = x^2. \tag{3.7.1}$$

We recall that $\beta = \tan\phi$, where the angle ϕ is defined in Figure 3.7.1. Note that the x^2 and y coordinates are the same. The inverse relations are

$$x^1 = x - \beta y, \qquad x^2 = y. \tag{3.7.2}$$

The covariant base vectors are given by

$$\mathbf{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{g}_2 = \begin{bmatrix} \beta \\ 1 \end{bmatrix} \tag{3.7.3}$$

and the contravariant base vectors are given by

$$\mathbf{g}^1 = \begin{bmatrix} 1 \\ -\beta \end{bmatrix}, \qquad \mathbf{g}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.7.4}$$

as shown in Figure 3.7.1.

3.7.1 Laplacian

The Laplacian of a function, $f(x^1, x^2)$, given in (3.6.13), simplifies to

$$\nabla^2 f = (1+\beta^2)\,\frac{\partial^2 f}{\partial x^1 \partial x^1} - 2\,\beta\,\frac{\partial^2 f}{\partial x^1 \partial x^2} + \frac{\partial^2 f}{\partial x^2 \partial x^2}, \tag{3.7.5}$$

which can be written as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^1 \partial x^1} + \Big(\frac{\partial}{\partial x^2} - \beta\,\frac{\partial}{\partial x^1}\Big)^2 f, \tag{3.7.6}$$

with the understanding that the square of the operator inside the parentheses expands as an operator.

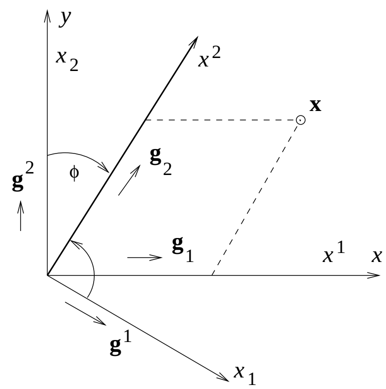

FIGURE 3.7.1 Illustration of two-dimensional oblique rectilinear coordinates in the canonical form.

3.7.2 Scaled coordinates

We may scale the second contravariant coordinate, x^2, and refer to the scaled coordinates, $\widetilde{x}^1$ and $\widetilde{x}^2$, defined by

$$x^1 = x - \beta y = \widetilde{x}^1, \qquad x^2 = y = \frac{1}{\sqrt{1+\beta^2}}\,\widetilde{x}^2. \tag{3.7.7}$$

Inverting these equations, we obtain

$$x = x^1 + \frac{\beta}{\sqrt{1+\beta^2}}\,\widetilde{x}^2, \qquad y = \frac{1}{\sqrt{1+\beta^2}}\,\widetilde{x}^2. \tag{3.7.8}$$

The expression for the Laplacian becomes

$$\nabla^2 f = (1+\beta^2)\Big(\frac{\partial^2 f}{\partial\widetilde{x}^1\partial\widetilde{x}^1} - 2\,\frac{\beta}{\sqrt{1+\beta^2}}\,\frac{\partial^2 f}{\partial\widetilde{x}^1\partial\widetilde{x}^2} + \frac{\partial^2 f}{\partial\widetilde{x}^2\partial\widetilde{x}^2}\Big). \tag{3.7.9}$$

Setting $\beta = \tan\phi$, we obtain

$$\nabla^2 f = \frac{1}{\cos^2\phi}\Big(\frac{\partial^2 f}{\partial\widetilde{x}^1\partial\widetilde{x}^1} - 2\,\sin\phi\,\frac{\partial^2 f}{\partial\widetilde{x}^1\partial\widetilde{x}^2} + \frac{\partial^2 f}{\partial\widetilde{x}^2\partial\widetilde{x}^2}\Big), \tag{3.7.10}$$

where the angle ϕ is defined in Figure 3.7.1.

3.7.3 Small deviations from Cartesian coordinates

Consider Poisson's equation,

$$\nabla^2 f + s = 0, \tag{3.7.11}$$

with reference to the Laplacian given in (3.7.5), where s is a specified source term. Now introducing a perturbation expansion for small β,

$$f = f_0 + \beta\, f_1 + \cdots\,, \tag{3.7.12}$$

we obtain the leading-order equation

$$\frac{\partial^2 f_0}{\partial x^1 \partial x^1} + \frac{\partial^2 f_0}{\partial x^2 \partial x^2} + s = 0, \tag{3.7.13}$$

and the first-order equation

$$\frac{\partial^2 f_1}{\partial x^1 \partial x^1} + \frac{\partial^2 f_1}{\partial x^2 \partial x^2} = 2\,\frac{\partial^2 f_0}{\partial x^1 \partial x^2}. \tag{3.7.14}$$

These two Poisson equations provide us with the first two leading-order solutions to be found subject to suitable boundary conditions.

3.7.4 Flow through an oblique rectangular pipe

As an example, we consider unidirectional viscous flow through a pipe whose cross-sectional shape is a parallelogram with four sides located at $x^1 = \pm a$ and $x^2 = \pm b$. The solution for the velocity satisfies the Poisson equation with a constant source term, s, subject to the no-slip boundary condition around the four edges, $f = 0$.

For convenience, we set $x^1 = \xi$ and $x^2 = \eta$. The zeroth-order solution is found readily as a Fourier series, as discussed in Section 5.1,

$$f_0(\xi, \eta) = \tfrac{1}{2}\, s\, b^2\, \Psi(\xi, \eta), \tag{3.7.15}$$

where

$$\Psi(\xi, \eta) = 1 - \frac{\eta^2}{b^2} + 4 \sum_{n=1}^{\infty} (-1)^n\, \frac{1}{\alpha_n^3} \frac{\cosh(\alpha_n\, \xi/b)}{\cosh(\alpha_n\, a/b)}\, \cos(\alpha_n \frac{\eta}{b}) \tag{3.7.16}$$

and $\alpha_n = (n - \frac{1}{2})\,\pi$. We find that

$$\frac{\partial^2 f_0}{\partial\xi\partial\eta} = -2\,\frac{s}{b^2}\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{\alpha_n}\frac{\sinh(\alpha_n\,\xi/b)}{\cosh(\alpha_n\,a/b)}\,\sin(\alpha_n\frac{\eta}{b}), \qquad (3.7.17)$$

which satisfies Laplace's equation in ξ and η. Substituting this expression into the right-hand side of (3.7.14) provides us with a Poisson equation for the first-order solution, f_1. The Poisson equation can be found by Fourier expansions or other numerical methods.

3.7.5 Finite-difference method

In canonical oblique coordinates, the Poisson equation,

$$\nabla^2 f + s = 0, \qquad (3.7.18)$$

takes the form

$$(1+\beta^2)\,\frac{\partial^2 f}{\partial\xi^2} - 2\,\beta\,\frac{\partial^2 f}{\partial\xi\partial\eta} + \frac{\partial^2 f}{\partial\eta^2} + s = 0, \qquad (3.7.19)$$

where s is a specified source term.

The solution can be found using a standard finite-difference method on a uniform grid with grid spacings $\Delta\xi$ and $\Delta\eta$ based on the finite-difference approximations

$$\frac{\partial^2 f}{\partial\xi^2} \simeq \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{\Delta\xi^2}, \qquad \frac{\partial^2 f}{\partial\eta^2} \simeq \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{\Delta\eta^2} \qquad (3.7.20)$$

for the pure second derivatives, and

$$\frac{\partial^2 f}{\partial\xi\partial\eta} \simeq \frac{1}{4\Delta\xi\Delta\eta}\,(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}) \qquad (3.7.21)$$

for the mixed second derivative.

The discretized Poisson equation (3.7.19) at the ij finite-difference node of a uniform Cartesian grid in the $\xi\eta$ plane takes the form

$$\begin{aligned}(1+\beta^2)\,(f_{i-1,j} - 2f_{i,j} + f_{i+1,j})&\\ -\tfrac{1}{2}\,\beta\,\varrho\,(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1})&\\ +\varrho^2\,(f_{i,j-1} - 2f_{i,j} + f_{i,j+1}) + \Delta\xi^2 s = 0,&\end{aligned} \tag{3.7.22}$$

where subscripts indicate grid values and

$$\varrho = \frac{\Delta\xi}{\Delta\eta} \tag{3.7.23}$$

is the ratio of the grid spacings.

Collecting all the unknown grid values and implementing the boundary conditions provides us with a system of linear equations,

$$\mathbf{A}\cdot\mathbf{f} = \mathbf{b}. \tag{3.7.24}$$

where $\mathbf{A}$ is a constructed coefficient matrix, $\mathbf{b}$ is a known right-hand side incorporating the boundary conditions, and the unknown grid values are collected into a long vector $\mathbf{f}$.

3.7.6 Construction of a linear system

A programmable algorithm can be designed for generating the coefficient matrix, $\mathbf{A}$, and right-hand side, $\mathbf{b}$, thereby circumventing the daunting task of manual bookkeeping. The main idea is to recast the system $\mathbf{A}\cdot\mathbf{f} = \mathbf{b}$ into the form

$$\mathbf{r} \equiv \mathbf{A}\cdot\mathbf{f} - \mathbf{b}, \tag{3.7.25}$$

where $\mathbf{r}$ is a residual vector, and then note that the matrix $\mathbf{A}$ contains the partial derivatives of the scalar components of the residual vector, $\mathbf{r}$, with respect to the components f_ℓ,

$$A_{k,\ell} = \frac{\partial r_k}{\partial f_\ell} = r_k(f_m = \delta_{m,\ell}) - r_k(\mathbf{f} = \mathbf{0}), \tag{3.7.26}$$

where $\delta_{m,\ell}$ is Kronecker's delta and the index m runs through the solution vector. To compute $\mathbf{A}$ and $\mathbf{b}$, we only require a subroutine or computer function that receives the components of the vector $\mathbf{f}$ and generates the vector $\mathbf{r}$. The method of impulses involves scanning sequentially all nodes while setting $f = 1$ at the current node and $f = 0$ at all other nodes,

The following Matlab function named *pois_fds_DDDD*, located in directory OBLIQUE of TUNLIB, computes the coefficient matrix, $\mathbf{A}$, and right-hand side, $\mathbf{b}$, when the solution domain is a rectangle in the $\xi\eta$ plane and the Dirichlet boundary condition is specified around the four sides of the solution domain:

```
function [mats,mat,rhs] = pois_fds_DDDD  ...
  ...
  (ax,bx ...
  ,ay,by ...
  ,beta ...
  ,Nx,Ny ...
  ,source ...
  ,w,q,z,v ...
  )

%===============================================
% Generate a finite-difference linear system
% for the Poisson equation in oblique coordinates, (x, y),
% inside a rectangle confined by ax<x<bx, ay<y<by
% where x = xi and y = eta
%
% Equation is:  Lalp(f) + source = 0
% System is :   mat * x = rhs
%
% Boundary conditions:   f = w   at x = ax
%                        f = q   at x = bx
%                        f = z   at y = ay
%                        f = v   at y = by
%
% The system is generated by the method of impulses
% which involves setting f=1 to one grid nodes
% while all other grid values are held at f=0
%
% SYMBOLS:
```

```
% -------
%
% Nx...   intervals in x direction
% Ny...   intervals in y direction
% mats... system size
% mat..   finite-difference matrix
% rhs..   right-hand side
%
% unknown vector is comprised of sequential values (i,j):
% (horizontal and then up)
%
%  2,2   3,2   4,2    ...  Nx-1,2   Nx,2
%  2,3   3,3   4,3    ...  Nx-1,3   Nx,2
%  ...
%
%  2,Ny  3,Ny  4,Ny   ...  Nx-1,Ny  Nx,Ny
%=====================================

%--------
% prepare
%--------

 Dx = (bx-ax)/Nx; Dy = (by-ay)/Ny;

 Dx2 = 2.0*Dx; Dy2 = 2.0*Dy;
 Dxs = Dx^2;   Dys = Dy^2;

 vp = Dx/Dy; vps = vp*vp;

 cf1 = 1.0+beta^2;
 cf2 =-0.5*beta*vp;

%-------------------
% initialize to zero
%-------------------

 for j=2:Ny
   for i=2:Nx
      f(i,j) = 0.0;
```

```
   end
 end

%------------------------------
% Dirichlet boundary conditions
%------------------------------

  for j=2:Ny
    f(   1,j) = w(j);     % left side
    f(Nx+1,j) = q(j);     % right side
  end

  for i=2:Nx+1
    f(i,   1) = z(i);     % down
    f(i,Ny+1) = v(i);     % up
  end

%---
% right-hand side
%---

 p = 0;     % counter

 for j=2:Ny
   for i=2:Nx
     p = p+1;
        R =            cf1*f(i+1,j) ...
          -  2.0*(cf1+vps)*f(i,j)...
          +            cf1*f(i-1,j)...
          +            vps*f(i,j-1)...
          +            vps*f(i,j+1)...
          + cf2*f(i+1,j+1) ...
          - cf2*f(i-1,j+1) ...
          - cf2*f(i+1,j-1) ...
          + cf2*f(i-1,j-1) ...
          + Dxs*source(i,j);
      rhs(p) = - R;
   end
 end
```

```
 mats = p;         % system size

%-------------------------------
% scan row-by-row to compute mat
%-------------------------------

      t = 0;    % counter

      for s=2:Ny
        for l=2:Nx

          t = t+1;
          f(l,s) = 1.0;  %  impulse

          p = 0;      % counter

          for j=2:Ny
            for i=2:Nx
             p = p+1;
             R =       cf1*f(i+1,j) ...
          -2.0*  (cf1+vps)*f(i,j)...
          +            cf1*f(i-1,j)...
          +            vps*f(i,j-1)...
          +            vps*f(i,j+1)...
          + cf2*f(i+1,j+1) ...
          - cf2*f(i-1,j+1) ...
          - cf2*f(i+1,j-1) ...
          + cf2*f(i-1,j-1) ...
          + Dxs*source(i,j);
             mat(p,t) = R+rhs(p);

            end
          end

          f(l,s) = 0.0;   % reset

        end
      end
```

```
%-----
% done
%-----

 return
```

3.7.7 Finite-difference code

The following Matlab code named *oblique*, located in directory OBLIQUE of TUNLIB, calls the function *pois_fds_DDDD*, solves the linear system, and displays the solution:

```
 phi = 0.25*pi;

 ax =-1.0; bx = 1.0;
 ay =-1.0; by = 1.0;
 Nx = 16;  Ny = 16;

 beta = tan(phi);

%--------------------
% boundary conditions
% and source term
%--------------------

  for j=1:Ny+1
    w(j) = 0.0;  % example
    q(j) = 0.0;  % example
  end

  for i=1:Nx+1
    z(i) = 0.0;  % example
    v(i) = 0.0;  % example
  end

  for i=1:Nx+1
    for j=1:Ny+1
      source(i,j) = 10.0; % example
    end
```

```
  end

%--------------------------
% generate the linear system
%--------------------------

  [mats,mat,rhs] = pois_fds_DDDD  ...
    ...
   (ax,bx ...
   ,ay,by ...
   ,beta ...
   ,Nx,Ny ...
   ,source ...
   ,w,q,z,v ...
   );

%---
% solution
%---

  sol = rhs/mat';

%---
% distribute the solution
%---

 p = 0;     % counter

 for j=2:Ny
   for i=2:Nx
     p = p+1;
     f(i,j) = sol(p);
   end
 end

 for j=1:Ny+1
    f(   1,j) = w(j);
    f(Nx+1,j) = q(j);
 end
```

```
 for i=1:Nx+1
    f(i,   1) = z(i);
    f(i,Ny+1) = v(i);
 end

%---
% physical grid
%---

  Dx = (bx-ax)/Nx; Dy = (by-ay)/Ny;
  for j=1:Ny+1
   for i=1:Nx+1
     xincl = ax + (i-1.0)*Dx;
     yincl = ay + (j-1.0)*Dy;
     xphys(i,j) = xincl + beta*yincl;
     yphys(i,j) = yincl;
   end
  end

%---
% plot
%---

surf(xphys,yphys,f)
```

Running the code generates the graphics shown in Figure 3.7.1. Physically, the graph shown in this picture is the velocity profile established inside a tube with oblique cross-section in unidirectional viscous flow or the shape of deformed membrane attached to an oblique tube.

Exercises

3.7.1 Derive expression (3.7.10).

3.7.2 Write a piece of code, continuing the code listed in the text, that computes the flow rate in viscous unidirectional flow through a tube whose cross-section is a parallelogram.

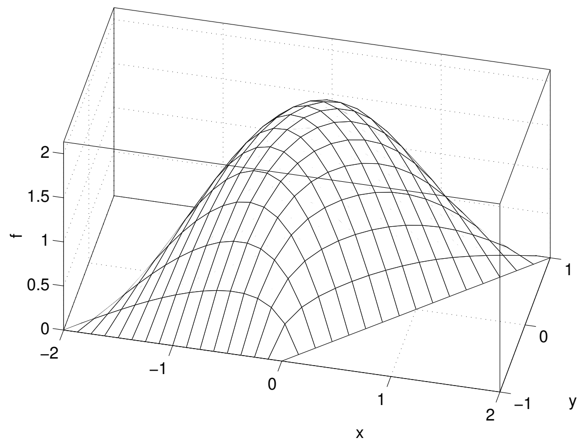

FIGURE 3.7.1 Solution of the Poisson equation with a uniform source term inside an inclined parallelogram computed with a finite-different method.

3.8 Channel coordinates

For convenience, we denote the contravariant coordinates as $\xi = x^1$ and $\eta = x^2$. The coordinate grid lines displayed in Figure 3.8.1(a, b) were generated using the mapping functions

$$x = L\xi, \qquad (3.8.1)$$
$$y = h\left((\eta - \tfrac{1}{2})\,(1 + w_-) + (\eta + \tfrac{1}{2})\,(1 + w_+) \right),$$

where L and h are arbitrary constants representing the channel length and semi-width,

$$w_-(\xi) = a_- \cos(2\pi\xi), \qquad w_+(\xi) = a_+ \cos(2\pi\xi) \qquad (3.8.2)$$

are the lower and upper wall profiles, and a_- and a_+ are the corresponding dimensionless amplitudes. Other more complicated upper and lower wall profiles can be chosen.

For the test section displayed in Figure 3.8.1 confined inside one period, the curvilinear coordinates ξ and η each takes values in the range $[-0.5, 0.5]$. The plots displayed in this figure, as well as all other

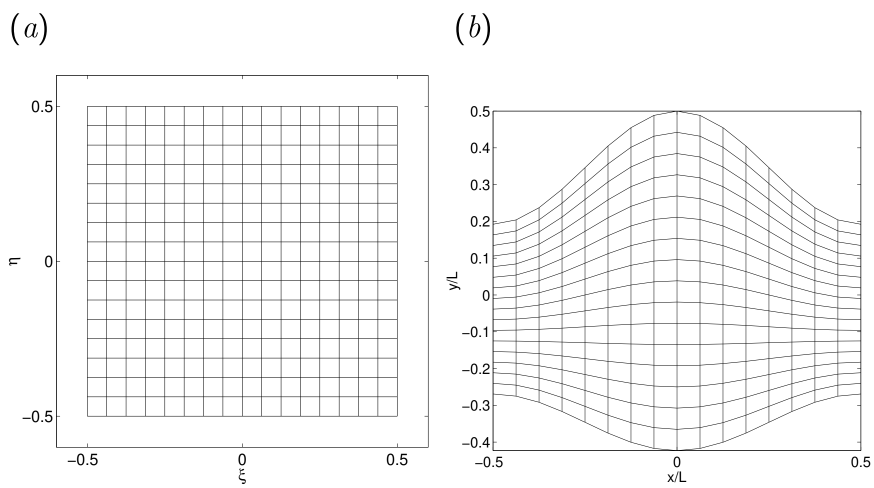

FIGURE 3.8.1 (a) Cartesian lines in a parametric square and (b) corresponding coordinate lines in a channel confined between two wavy walls for channel semi-width $h/L = 1$ and lower and upper wall amplitudes $a_-/L = 0.1$ and $a_+/L = 0.2$.

plots displayed in this section, were generated using a code named *poisson* located in directory CHANNEL of TUNLIB.

3.8.1 Base vectors and metric coefficients

We find by straightforward differentiation that the covariant base vectors are given by

$$\mathbf{g}_\xi \equiv \frac{\partial \mathbf{x}}{\partial \xi} = \begin{bmatrix} L \\ h\,\big(\,(\eta - \tfrac{1}{2})\,w'_- + (\eta + \tfrac{1}{2})\,w'_+\,\big) \end{bmatrix} \qquad (3.8.3)$$

and

$$\mathbf{g}_\eta \equiv \frac{\partial \mathbf{x}}{\partial \eta} = \begin{bmatrix} 0 \\ h\,\big((1 + w_-) + (1 + w_+)\,\big) \end{bmatrix}, \qquad (3.8.4)$$

where a prime denotes a derivative with respect to ξ. These two covariant vectors are mutually perpendicular only at the planes of symmetry positioned at $x/L = -0.5, 0, 0.5$, as shown in Figure 3.8.2(a). The

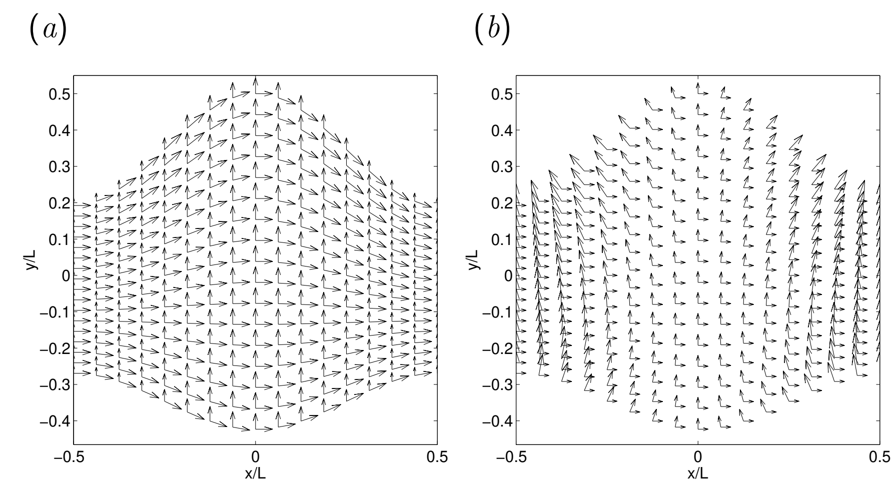

FIGURE 3.8.2 (a) Covariant and (b) contravariant base vector fields in a channel-like domain.

associated contravariant base vectors are shown in Figure 3.8.2(b). Cursory inspection confirms the biorthogonality of the covariant and contravariant sets of base vectors at every grid node.

The covariant metric coefficients are given by

$$\begin{aligned} g_{\xi\xi} &= L^2 + h^2 \left((\eta - \tfrac{1}{2})\, w'_- + (\eta + \tfrac{1}{2})\, w'_+ \right)^2, \\ g_{\eta\eta} &= \left((1 + w_-) + (1 + w_+) \right)^2, \\ g_{\xi\eta} &= g_{\eta\xi} = h^2 \left((\eta - \tfrac{1}{2})\, w'_- + (\eta + \tfrac{1}{2})\, w'_+ \right) \left((1 + w_-) + (1 + w_+) \right). \end{aligned} \tag{3.8.5}$$

The determinant $g = \det(\mathbf{g})$ depends on ξ but not on η, as shown in Figure 3.8.3(a).

3.8.2 Laplacian

The Laplacian of a function, $f(x, y)$ was given in equation (3.5.30), repeated below for convenience,

$$\nabla^2 f = g^{\xi\xi} \frac{\partial^2 f}{\partial \xi^2} + 2 g^{\xi\eta} \frac{\partial^2 f}{\partial \xi \partial \eta} + g^{\eta\eta} \frac{\partial^2 f}{\partial \eta^2} + v^{\xi} \frac{\partial f}{\partial \xi} + v^{\eta} \frac{\partial f}{\partial \eta}, \tag{3.8.6}$$

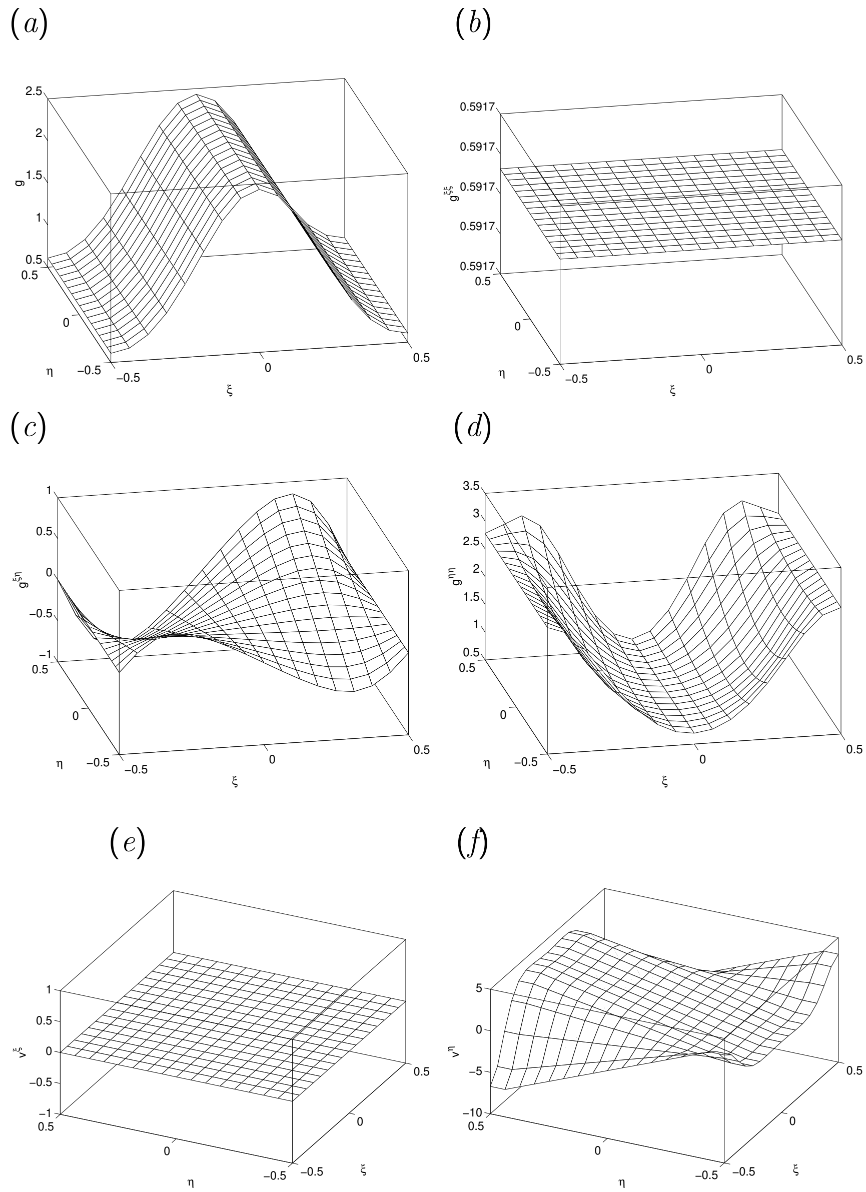

FIGURE 3.8.3 Distribution of (a) the determinant of the matrix of covariant metric coefficients, g, and $(b\text{-}d)$ contravariant metric coefficients, $g^{\xi\xi}$, $g^{\xi\eta}$, and $g^{\eta\eta}$, for channel coordinates. $(e,\ f)$ Distribution of the coefficients v^{ξ} and v^{η} defined in (3.8.7).

where

$$v^k \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \Big(g^{ki} \sqrt{g} \Big). \tag{3.8.7}$$

The distributions of the contravariant metric coefficients, $g^{\xi\xi}$, $g^{\xi\eta}$, and $g^{\eta\eta}$, are shown in Figure 3.8.3(b–d). Note that the distribution of $g^{\xi\xi}$ shown in Figure 3.8.3(b) is flat. The distributions of the coefficients v^{η} and v^{η} are plotted in Figure 3.8.3(e, f).

3.8.3 Poisson equation

Now we consider the Poisson equation in the xy plane,

$$\nabla^2 f + s = 0, \tag{3.8.8}$$

where s is a specified source distribution. Substituting the expression for the Laplacian, we obtain

$$g^{\xi\xi} \frac{\partial^2 f}{\partial \xi^2} + 2 g^{\xi\eta} \frac{\partial^2 f}{\partial \xi \partial \eta} + g^{\eta\eta} \frac{\partial^2 f}{\partial \eta^2} + v^{\xi} \frac{\partial f}{\partial \xi} + v^{\eta} \frac{\partial f}{\partial \eta} + s(\xi, \eta) = 0. \tag{3.8.9}$$

The solution can be found using a standard finite-difference method on a uniform grid with grid spacings $\Delta\xi$ and $\Delta\eta$ based on second-order finite-difference approximations at the i, j node. The approximations are

$$\frac{\partial f}{\partial \xi} \simeq \frac{1}{2\Delta\xi} (f_{i+1,j} + f_{i-1,j}), \qquad \frac{\partial f}{\partial \eta} \simeq \frac{1}{2\Delta\eta} (f_{i,j+1} + f_{i,j-1}) \tag{3.8.10}$$

for the first derivatives,

$$\frac{\partial^2 f}{\partial \xi^2} \simeq \frac{1}{\Delta\xi^2} (f_{i-1,j} - 2 f_{i,j} + f_{i+1,j}), \qquad \frac{\partial^2 f}{\partial \eta^2} \simeq \frac{1}{\Delta\eta^2} (f_{i,j-1} - 2 f_{i,j} + f_{i,j+1}) \tag{3.8.11}$$

for the pure second derivatives, and

$$\frac{\partial^2 f}{\partial\xi\partial\eta} \simeq \frac{1}{4\Delta\xi\Delta\eta}\,(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}) \tag{3.8.12}$$

for the mixed second derivative at the i, j node, where the indices i and j are grid-node labels. Note that the central value, $f_{i,j}$, appears only in the pure second derivatives.

The discretized Poisson equation (3.8.6) at the ij finite-difference node of a uniform Cartesian grid takes the form

$$\begin{aligned} & g^{\xi\xi}\,(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \\ & \quad + \tfrac{1}{2}\, g^{\xi\eta}\mu\,(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}) \\ & \qquad + \tfrac{1}{2}\,\Delta\xi\,\Big(v^{\xi}(f_{i+1,j} - f_{i-1,j}) + \mu v^{\eta}(f_{i,j+1} - f_{i,j-1})\Big) \\ & \qquad\quad + \mu^2\, g^{\eta\eta}\,(f_{i,j-1} - 2f_{i,j} + f_{i,j+1}) + \Delta\xi^2\, s = 0, \end{aligned} \tag{3.8.13}$$

where

$$\mu = \frac{\Delta\xi}{\Delta\eta} \tag{3.8.14}$$

is the ratio of the grid spacings and subscripts indicate grid values.

3.8.4 Linear equations

Collecting all unknown grid values into a long vector, $\mathbf{f}$, provides us with the system of linear equations,

$$\mathbf{A}\cdot\mathbf{f} = \mathbf{b}. \tag{3.8.15}$$

where $\mathbf{A}$ is a coefficient matrix and $\mathbf{b}$ is a known right-hand side incorporating the boundary conditions. The formulation is similar to that discussed in Section 3.6 for solving the Poisson equation in oblique rectilinear coordinates.

3.8.5 Construction of the coefficient matrix for PPDD boundary conditions

The matrix, $\mathbf{A}$, and right-hand side, $\mathbf{b}$, can be constructed by the method of impulses, which involves scanning sequentially all nodes while setting $f = 1$ at the current node and $f = 0$ at all other nodes, as discussed in Section 3.6.

A Matlab function named *pois_fds_PPDD*, located in directory CHANNEL of TUNLIB, computes the coefficient matrix, $\mathbf{A}$, and right-hand side, $\mathbf{b}$. The periodicity condition (PP) is imposed along the ξ axis and the Dirichlet boundary condition (DD) is imposed around the top and bottom sides of the solution domain. The input to this function includes three contravariant metric coefficients and the nodal values of v^ξ and v^η:

```
function [mats,mat,rhs] = pois_fds_PPDD  ...
  ...
   (ax,bx ...
   ,ay,by ...
   ,Nx,Ny ...
   ,g11,g12,g22 ...
   ,v1,v2 ...
   ,source ...
   ,z,v ...
   )

%===============================================
% Generate a finite-difference linear system
% for the Poisson equation
% inside a rectangle confined in
% ax<x<bx, ay<y<by
%
% Equation is:    Lalp(f) + source = 0
%
% System is :     mat * x = rhs
%
% Boundary conditions:   f = periodic   at x = ax
%                        f = periodic   at x = bx
%                        f = z          at y = ay
```

```
%                        f = v          at y = by
%
% The system is generated by the method of impulses
%
% SYMBOLS:
% -------
%
% Nx...   intervals in x direction
% Ny...   intervals in y direction
% mats... system size
% mat..   finite-difference matrix
% rhs..   right-hand side
%
% unknown vector is comprised of sequential values (i,j):
% (horizontal and then up)
%
%  2,2   3,2   4,2    ...  Nx,2   Nx+1,2
%  2,3   3,3   4,3    ...  Nx,3   Nx+1,2
%  ...
%
%  2,Ny  3,Ny  4,Ny   ...  Nx,Ny  Nx+1,Ny
%=====================================

%-------------
% preparations
%-------------

 Dx = (bx-ax)/Nx;
 Dy = (by-ay)/Ny;

%------------------
% more preparations
%------------------

 Dx2 = 2.0*Dx;
 Dy2 = 2.0*Dy;

 Dxs = Dx^2;
 Dys = Dy^2;
```

```
 vp  = Dx/Dy;
 vps = vp*vp;

%------------------
% initialize to zero
%------------------

 for j=1:Ny+1
   for i=1:Nx+2
     f(i,j) = 0.0;
   end
 end

%-----------------------------
% Dirichlet boundary conditions
% at bottom and top
%-----------------------------

  for i=1:Nx+2
    f(i,   1) = z(i);     % bottom
    f(i,Ny+1) = v(i);     % top
  end

%---
% wrap
%---

  for j=1:Ny+1
    f(1,j) = f(Nx+1,j); f(Nx+2,j) = f(2,j);
  end

%---
% right-hand side
%---

 p = 0;     % counter
 for j=2:Ny
   for i=2:Nx+1
```

```
     p = p+1;
     R =       g11(i,j)*(f(i-1,j)-2.0*f(i,j)+f(i+1,j)) ...
         + vps*g22(i,j)*(f(i,j-1)-2.0*f(i,j)+f(i,j+1)) ...
         + 0.5*vp*g12(i,j)*(f(i+1,j+1)-f(i-1,j+1) ...
                         -f(i+1,j-1)+f(i-1,j-1)) ...
         + 0.5*Dx*   v1(i,j)*(f(i+1,j)-f(i-1,j)) ...
         + 0.5*Dx*vp*v2(i,j)*(f(i,j+1)-f(i,j-1)) ...
         + Dxs*source(i,j);
      rhs(p) = - R;
   end
 end

 mats = p;         % system size

%-------------------------------
% scan row-by-row to compute mat
%-------------------------------

      t = 0;    % counter

      for s=2:Ny
        for l=2:Nx+1

          t = t+1;
          f(l,s) = 1.0;  %  impulse

          for j=2:Ny   % wrap
           f(1,j) = f(Nx+1,j);
           f(Nx+2,j) = f(2,j);
          end

          p = 0;      % counter

          for j=2:Ny
            for i=2:Nx+1
             p = p+1;
             R =     g11(i,j)*(f(i-1,j)-2.0*f(i,j)+f(i+1,j)) ...
               + vps*g22(i,j)*(f(i,j-1)-2.0*f(i,j)+f(i,j+1)) ...
               + 0.5*vp*g12(i,j)*(f(i+1,j+1)-f(i-1,j+1) ...
```

```
                                  -f(i+1,j-1)+f(i-1,j-1)) ...
              + 0.5*Dx*   v1(i,j)*(f(i+1,j)-f(i-1,j)) ...
              + 0.5*Dx*vp*v2(i,j)*(f(i,j+1)-f(i,j-1)) ...
              + Dxs*source(i,j);
              mat(p,t) = R+rhs(p);
           end
         end

         f(1,s) = 0.0;   % reset

         for j=2:Ny  % wrap
           f(1,j) = f(Nx+1,j);
           f(Nx+2,j) = f(2,j);
         end

       end
     end

%-----
% done
%-----

 return
```

3.8.6 Finite-difference code

The following Matlab code named *channel*, located in directory CHANNEL of TUNLIB, calls the preceding function, solves the linear system of equations for the nodal values, and visualizes the solution.

```
%=============================
% Solution of the Poisson equation
% with Dirichlet boundary conditions
% in curvilinear coordinates
% in a channel-like domain
%
% xi = x^1, eta =x^2
%
% xi varies in [ax, bx]
% eta varies in [ay, by]
```

```
%============================

%---
% channel geometrical parameters
%---

ichannel = 2; % annulus
ichannel = 1; % channel

L = 1.3; h = 0.45;

%---
% amplitude and phase
%---

ahig = 0.2; alow = 0.1;

phasehig = 0.25*pi;
phasehig = 0.0*pi;

%---
% parameters
%---

N1 = 1*16;
N2 = 1*16;

if(ichannel==2)
 N1 = 2*16;
 N2 = 1*16;
end

a1 = -0.5; b1 = 0.5;
a2 = -0.5; b2 = 0.5;

%---
% prepare
%---
```

```
Dx1 = (b1-a1)/N1;
Dx2 = (b2-a2)/N2;

for i=1:N1+2
  xdiv1(i) = a1+(i-1.0)*Dx1;
end

for j=1:N2+2
  xdiv2(j) = a2+(j-1.0)*Dx2;
end

for j=1:N2+2
 for i=1:N1+2
  x1(i,j) = xdiv1(i);
  x2(i,j) = xdiv2(j);
 end
end

%---
% grid in the xy plane
%---

cf1 = 0.0;
cf2 = 0.0;

for j=1:N2+2
 for i=1:N1+2

  x(i,j) = L*x1(i,j);
  wlow(i) = h + alow*(cos(2*pi*x1(i,j)) ...
             +   cf1*cos(4*pi*x1(i,j)) ...
             +   cf2*cos(6*pi*x1(i,j)) );
  whig(i) = h+ahig*cos(2*pi*x1(i,j)+phasehig);
  y(i,j) = (x2(i,j)-0.5)*wlow(i) + (x2(i,j)+0.5)*whig(i);

  if(ichannel==2) % annulus

   im = sqrt(-1);
   r = exp(1.0+y(i,j)/h);
```

```
   theta = 2*pi*x(i,j)/L;
   x(i,j) = r*cos(theta);
   y(i,j) = r*sin(theta);

  end

 end
end

%---
% covariant base vectors
%---

for j=2:N2+1
 for i=2:N1+1
  gcov1_x(i,j) = (x(i+1,j)-x(i-1,j))/(2.0*Dx1);
  gcov1_y(i,j) = (y(i+1,j)-y(i-1,j))/(2.0*Dx1);
  gcov2_x(i,j) = (x(i,j+1)-x(i,j-1))/(2.0*Dx2);
  gcov2_y(i,j) = (y(i,j+1)-y(i,j-1))/(2.0*Dx2);
 end
end

for i=2:N1+1
  gcov1_x(i,1) = (x(i+1,1)-x(i-1,1))/(2.0*Dx1);
  gcov1_y(i,1) = (y(i+1,1)-y(i-1,1))/(2.0*Dx1);
  gcov2_x(i,1) = (x(i,2)-x(i,1))/Dx2;
  gcov2_y(i,1) = (y(i,2)-y(i,1))/Dx2;
end

%---
% wrap
%---

for j=1:N2+1
  gcov1_x(1,j) = gcov1_x(N1+1,j);
  gcov1_y(1,j) = gcov1_y(N1+1,j);
  gcov2_x(1,j) = gcov2_x(N1+1,j);
  gcov2_y(1,j) = gcov2_y(N1+1,j);
end
```

```
%---
% covariant metric coefficients
%---

for j=1:N2+1
 for i=1:N1+1
  covmet11(i,j) = gcov1_x(i,j)*gcov1_x(i,j) ...
                + gcov1_y(i,j)*gcov1_y(i,j);
  covmet12(i,j) = gcov1_x(i,j)*gcov2_x(i,j) ...
                + gcov1_y(i,j)*gcov2_y(i,j);
  covmet21(i,j) = covmet12(i,j);
  covmet22(i,j) = gcov2_x(i,j)*gcov2_x(i,j) ...
                + gcov2_y(i,j)*gcov2_y(i,j);
  covg(i,j) = covmet11(i,j)*covmet22(i,j)-covmet12(i,j)^2;

  srcovg(i,j) = sqrt(covg(i,j));

  covmet = [ covmet11(i,j), covmet12(i,j);...
             covmet21(i,j), covmet22(i,j)];
  invcovmet = inv(covmet);

  conmet11(i,j) = invcovmet(1,1);
  conmet12(i,j) = invcovmet(1,2);
  conmet21(i,j) = invcovmet(2,1);
  conmet22(i,j) = invcovmet(2,2);
  cong(i,j) = conmet11(i,j)*conmet22(i,j)-conmet12(i,j)^2;


 end
end

%---
% wrap
%---

for j=1:N2+1
 covmet11(N1+2,j) = covmet11(2,j);
 covmet12(N1+2,j) = covmet12(2,j);
```

```
 covmet21(N1+2,j) = covmet21(2,j);
 covmet22(N1+2,j) = covmet22(2,j);
     covg(N1+2,j) =     covg(2,j);
   srcovg(N1+2,j) =   srcovg(2,j);
 conmet11(N1+2,j) = conmet11(2,j);
 conmet12(N1+2,j) = conmet12(2,j);
 conmet21(N1+2,j) = conmet21(2,j);
 conmet22(N1+2,j) = conmet22(2,j);
     cong(N1+2,j) =     cong(2,j);
end

%---
% contravariant base vectors
%---

for j=1:N2+1
 for i=1:N1+1
  gcon1_x(i,j) = ( covmet22(i,j)*gcov1_x(i,j) ...
                  -covmet12(i,j)*gcov2_x(i,j))/covg(i,j);
  gcon1_y(i,j) = ( covmet22(i,j)*gcov1_y(i,j) ...
                  -covmet12(i,j)*gcov2_y(i,j))/covg(i,j);
  gcon2_x(i,j) = (-covmet12(i,j)*gcov1_x(i,j) ...
                  +covmet11(i,j)*gcov2_x(i,j))/covg(i,j);
  gcon2_y(i,j) = (-covmet12(i,j)*gcov1_y(i,j) ...
                  +covmet11(i,j)*gcov2_y(i,j))/covg(i,j);
 end
end

%---
% wrap
%---

for j=1:N2+1
 gcon1_x(N1+2,j) = gcon1_x(2,j);
 gcon1_y(N1+2,j) = gcon1_y(2,j);
 gcon2_x(N1+2,j) = gcon2_x(2,j);
 gcon2_y(N1+2,j) = gcon2_y(2,j);
end
```

```
%-------------
% compute the effective velocities v1 and v2
%-------------

for j=2:N2
 for i=2:N1+1
  v1(i,j) = (conmet11(i+1,j)*srcovg(i+1,j) ...
            -conmet11(i-1,j)*srcovg(i-1,j))/(2.0*Dx1) ...
           +(conmet12(i,j+1)*srcovg(i,j+1) ...
            -conmet12(i,j-1)*srcovg(i,j-1))/(2.0*Dx2);
  v2(i,j) = (conmet21(i+1,j)*srcovg(i+1,j) ...
            -conmet21(i-1,j)*srcovg(i-1,j))/(2.0*Dx1) ...
           +(conmet22(i,j+1)*srcovg(i,j+1) ...
            -conmet22(i,j-1)*srcovg(i,j-1))/(2.0*Dx2);
  v1(i,j) = v1(i,j)/srcovg(i,j);
  v2(i,j) = v2(i,j)/srcovg(i,j);
 end
end

%---
% wrap
%---

for j=2:N2
  v1(1,j) = v1(N1+1,j);
  v2(1,j) = v2(N1+1,j);
end

for i=1:N1+1
  v1(i,1) = v1(i,2);
  v2(i,1) = v2(i,2);
  v1(i,N2+1) = v1(i,N2);
  v2(i,N2+1) = v2(i,N2);
end

%---------
% solve the Poisson equation
%---------
```

```
for j=1:N2+1
 for i=1:N1+2
  source(i,j) = 0.0;
  if(ichannel==1)
    source(i,j) = 5.0; % typical
  elseif(ichannel==2)
    source(i,j) = 1.0; % typical
  end
 end
end

for i=1:N1+2
  z(i) = 0.0; % typical
  v(i) = 0.1; % typical
  z(i) = 0.5; % typical
  v(i) = 0.0; % typical
end

%---
% generate the linear system
%---

[mats,mat,rhs] = pois_fds_PPDD  ...
  ...
   (a1,b1 ...
   ,a2,b2 ...
   ,N1,N2 ...
   ,conmet11,conmet12,conmet22 ...
   ,v1,v2 ...
   ,source ...
   ,z,v ...
   );

%---
% solution
%---

  sol = rhs/mat';
```

```
%---
% distribute the solution
%---

 p = 0;     % counter

 for j=2:N2
   for i=2:N1+1
     p = p+1;
     f(i,j) = sol(p);
   end
 end

  for i=1:N1+2  % boundary conditions
    f(i,   1) = z(i);
    f(i,N2+1) = v(i);
  end

  for j=1:N2+1   % wrap
    f(1,j) = f(N1+1,j);
  end

%---
% plot
%---

mesh(x(1:N1+1,1:N2+1),y(1:N1+1,1:N2+1),f(1:N1+1,1:N2+1))
```

Running the code for a particular set of conditions generates the graphics shown in Figure 3.8.4. Physically, the solution represents the velocity distribution in viscous unidirectional flow between two wavy walls inside a periodic channel.

3.8.7 Further channel-like geometries

Other wall geometries can be considered, as shown in Figure 3.8.5(a, b) for a domain confined between a lower doubly sinusoidal wall and an upper flat wall. Wrapping one period of the channel and stapling the ends, we obtain an annular domain between two generally eccentric

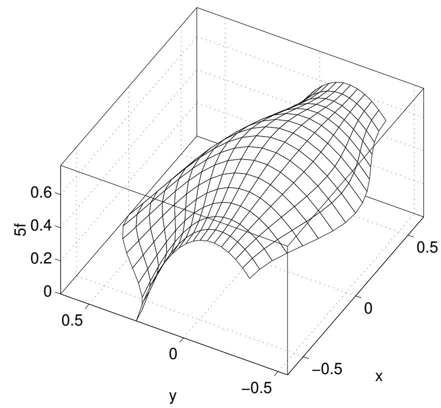

FIGURE 3.8.4 Solution of the Poisson equation with a uniform source term in a channel-like domain.

cylinders, as shown in Figure 3.8.5(c, d). The method is implemented in the code listed previously in this section.

Exercise

3.8.1 Reproduce Figures 3.8.1–3.8.4 for phase shift $\frac{1}{2}\pi$ between the upper and lower wall and discuss the results.

3.9 Inside a quadrilateral

The interior of a quadrilateral in the xy plane can be mapped to a rectangle in the parametric $\xi\eta$ plane confined inside $a_1 \le \xi \le b_1$ and $a_2 \le \eta \le b_2$, as shown in Figure 3.9.1.

3.9.1 Mapping

The mapping is mediated by the bilinear transformation

$$\mathbf{x} = \sum_{i=1}^{4} \phi_i(\xi, \eta)\, \mathbf{v}_i, \tag{3.9.1}$$

where $\mathbf{v}_i$ are the vertices of the quadrilateral in the xy plane. The

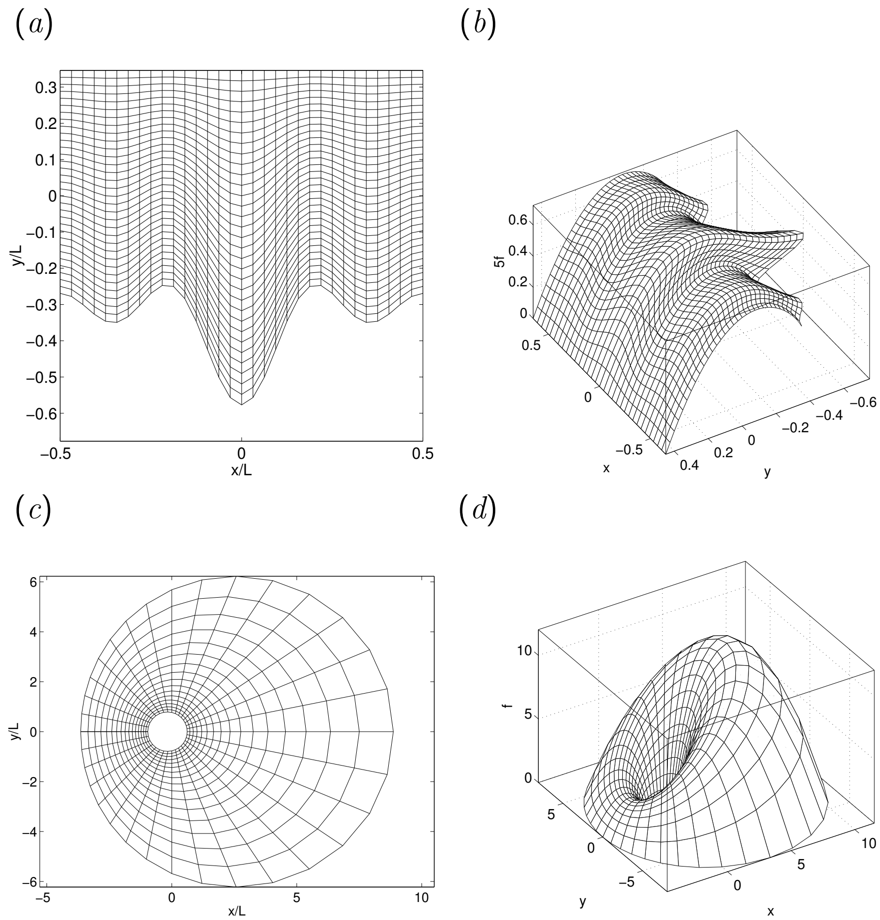

FIGURE 3.8.5 (a) Curvilinear grid and (b) solution of the Poisson equation with a uniform source term above a wall whose profile is described by two superposed sinusoids. (c) Curvilinear grid and (d) solution of the Poisson equation with a uniform source term inside an annulus.

corresponding interpolation functions are given by

$$\phi_1(\xi,\eta) = \frac{1}{A}\,(b_1-\xi)(b_2-\eta), \qquad \phi_2(\xi,\eta) = \frac{1}{A}\,(\xi-a_1)(b_2-\eta),$$
$$\phi_3(\xi,\eta) = \frac{1}{A}\,(\xi-a_1)(\eta-a_2), \qquad \phi_4(\xi,\eta) = \frac{1}{A}\,(b_1-\xi)(\eta-a_2), \tag{3.9.2}$$

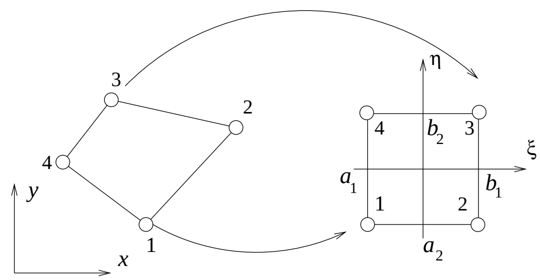

FIGURE 3.9.1 Mapping of a quadrilateral in the xy plane to a rectangle in the $\xi\eta$ plane by a bilinear transformation.

where $A = (b_1 - a_1)(b_2 - a_2)$. Vertex 1 is mapped to $\xi = a_1, \eta = a_2$, vertex 2 to $\xi = b_1, \eta = a_2$, vertex 3 to $\xi = b_1, \eta = b_2$, and vertex 4 to $\xi = a_1, \eta = b_2$. An edge of the quadrilateral in the xy plane is mapped to an edge in the $\xi\eta$ plane.

A typical example is shown in Figure 3.9.2(a, b). The distribution of the covariant and contravariant base vectors inside a quadrilateral is shown in Figure 3.9.2(c, d).

The distribution of the determinant $g = \det(\mathbf{g})$ is shown in Figure 3.9.3(a). The distributions of the contravariant metric coefficients, $g^{\xi\xi}$, $g^{\xi\eta}$, and $g^{\eta\eta}$, are shown in Figure 3.9.3(b–d).

The distributions of the coefficients v^η and v^η involved in the expression for the Laplacian shown in (3.8.9) are displayed in Figure 3.9.3(e, f).

3.9.2 Inverse mapping

To find the values of ξ and η corresponding to a specified point, $\mathbf{x}$, we may solve a system of two quadratic equations for two unknowns that arises from the x and y component of (3.9.1).

Alternatively and more efficiently, we note that each interpolation function has the form of an incomplete quadratic polynomial in two

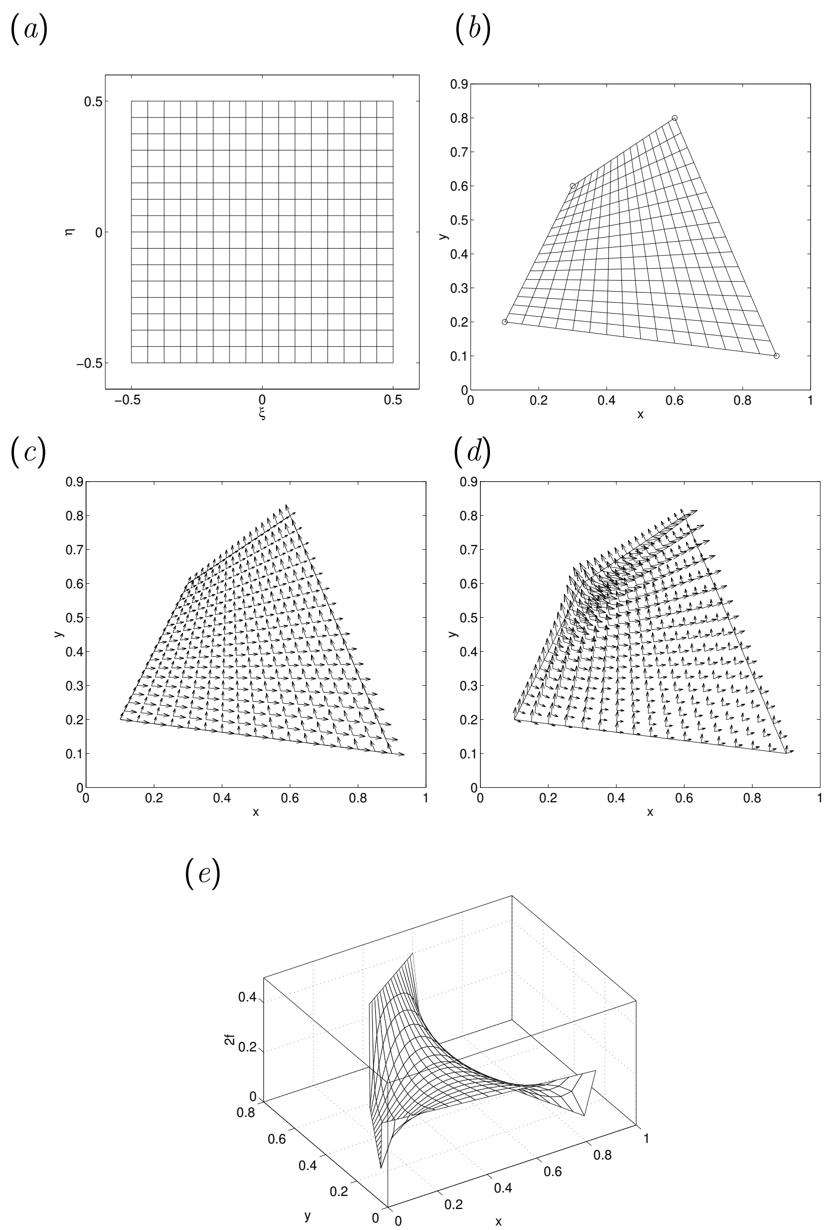

FIGURE 3.9.2 (a, b) Curvilinear coordinates generated by a bilinear transformation inside a quadrilateral defined by four arbitrary vertices in the xy plane. (c, d) Covariant and contravariant base vectors. (e) Solution of the Poisson equation with a uniform source term and Dirichlet boundary conditions.

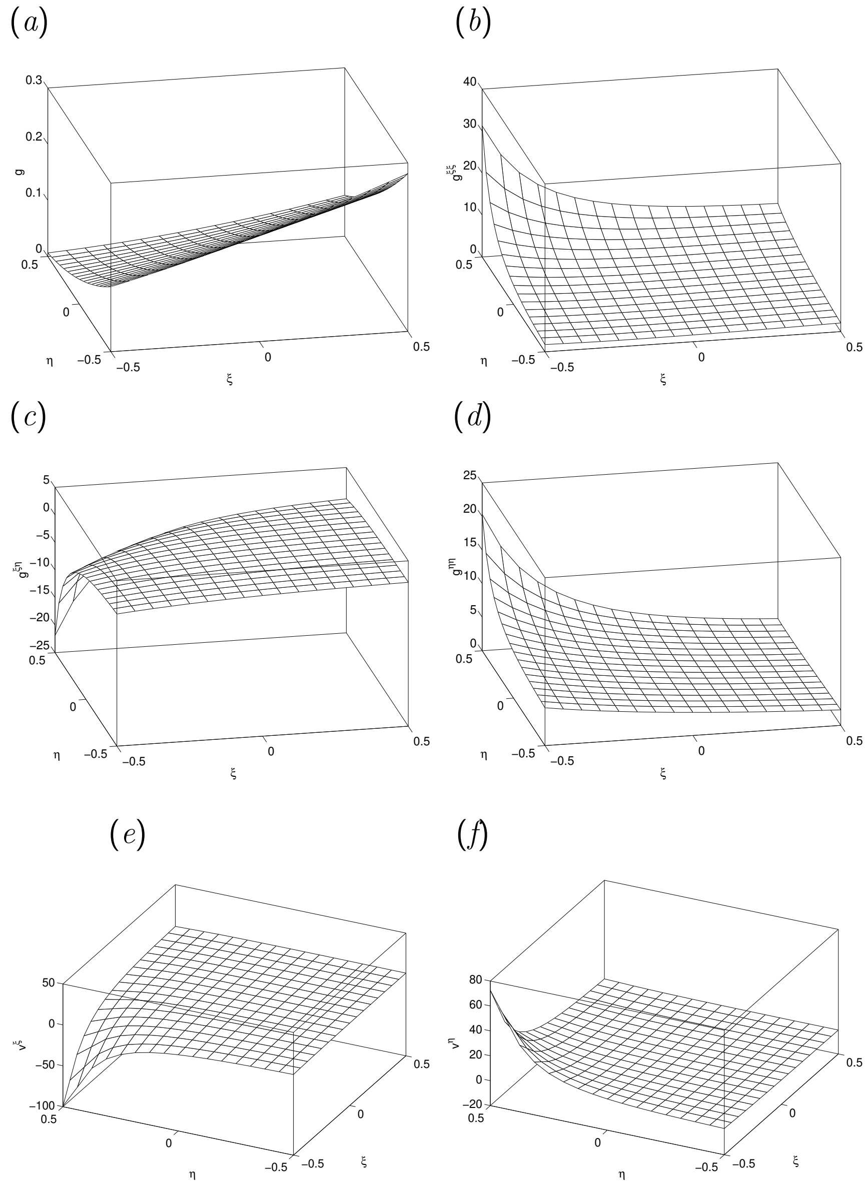

FIGURE 3.9.3 Distribution of (a) the determinant of the matrix of covariant coefficients, g, and $(b$-$d)$ contravariant metric coefficients inside a quadrilateral. $(e,\, f)$ Distribution of the coefficients v^ξ and v^η defined in (3.5.29).

variables missing the pure quadratic terms,

$$\phi_i = \kappa_i + \lambda_i \xi + \mu_i \eta + \nu_i \xi \eta, \tag{3.9.3}$$

where κ_i–ν_i are four constants. Correspondingly, we may write

$$\begin{aligned} x &= \mathcal{A}_x + \mathcal{B}_x \xi + \mathcal{C}_x \eta + \mathcal{D}_x \xi \eta, \\ y &= \mathcal{A}_y + \mathcal{B}_y \xi + \mathcal{C}_y \eta + \mathcal{D}_y \xi \eta, \end{aligned} \tag{3.9.4}$$

where $\mathcal{A}_x$–$\mathcal{D}_x$ is a set of four constants and $\mathcal{A}_y$–$\mathcal{D}_y$ is another set of constants. Each set of coefficients can be found by solving a system of four linear equations that arises by applying one of equations (3.9.4) at the four vertices, yielding

$$\begin{bmatrix} 1 & a_1 & a_2 & a_1 a_2 \\ 1 & b_1 & a_2 & b_1 a_2 \\ 1 & b_1 & b_2 & b_1 b_2 \\ 1 & a_1 & b_2 & a_1 b_2 \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_x \\ \mathcal{B}_x \\ \mathcal{C}_x \\ \mathcal{D}_x \end{bmatrix} = \begin{bmatrix} (v_1)_x \\ (v_2)_x \\ (v_3)_x \\ (v_4)_x \end{bmatrix}, \tag{3.9.5}$$

and a companion linear system where x is replaced by y. Combining the two equations in (3.9.4), we obtain a linear equation involving ξ and η,

$$\begin{aligned} \mathcal{D}_y x - \mathcal{D}_x y = {} & \mathcal{D}_y (\mathcal{A}_x + \mathcal{B}_x \xi + \mathcal{C}_x \eta) \\ & - \mathcal{D}_x (\mathcal{A}_y + \mathcal{B}_y \xi + \mathcal{C}_y \eta + \mathcal{D}_y \xi \eta). \end{aligned} \tag{3.9.6}$$

Solving this equation for η or ξ and substituting the result into one of the equations in (3.9.4) provides us with a quadratic equation for ξ or η.

3.9.3 *Poisson equation*

A finite-difference method for solving the Poisson equation inside a quadrilateral can be developed working as in Section 3.7 for a channel-like domain. A Matlab code entitled *quad*, located in directory QUAD of TUNLIB, solves the Poisson subject to the Dirichlet boundary condition around the four edges, and visualizes the solution.

Running the code for a particular set of conditions generates the graphics shown in Figure 3.9.2(e). Physically, the solution represents the velocity profile in viscous unidirectional flow or the shape of a membrane attached to the edges of the quadrilateral, subject to a pressure difference driving the the deformation.

3.9.4 Flow rate

The integral of the solution over the quadrilateral can be computed from the expression

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint f(\xi,\eta)\,\sqrt{g}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.9.7}$$

The integral on the right-hand side can be computed by standard numerical methods as a sum over cells defined by adjacent pairs of curvilinear grid lines. The cell values can be computed from the grid node values by sensible interpolation. In the case of unidirectional viscous flow, this integral represents the flow rate.

Exercise

3.9.1 Write a code that generates ξ and η for given x and y. If (ξ,η) lies inside the square, then (x,y) lies inside the quadrilateral.

3.10 Conformal mapping

Earlier in this chapter, we discussed numerical solutions of the Poisson equation in a channel-like or quadrilateral shaped domain in the xy plane by mapping each domain to a rectangle or square in the $\xi\eta$ plane, and then solving a modified Poisson equation inside the rectangle or square. An important advantage of orthogonal mapping is that it provides us with orthogonal curvilinear coordinates that simplify the modified Poisson equation and facilitate the implementation of numerical methods.

3.10.1 Mapping function

A convenient method of generating orthogonal coordinates is based on the notion of conformal mapping. To formulate the method, we introduce a complex variable in the xy plane,

$$z = x + \mathrm{i}y, \tag{3.10.1}$$

where i is the imaginary unit, and introduce another complex variable,

$$\zeta = \xi + \mathrm{i}\,\eta. \tag{3.10.2}$$

A point in the complex z plane can be mapped to a point in the complex ζ plane, and *vice versa*, using forward or reverse mapping,

$$z = \mathcal{F}(\zeta), \tag{3.10.3}$$

so that

$$x = \mathcal{F}_{\mathrm{real}}(\xi, \eta), \qquad y = \mathcal{F}_{\mathrm{imaginary}}(\xi, \eta). \tag{3.10.4}$$

It can be shown that, conformal mapping generates curvilinear coordinates that are orthogonal and isometric,

$$g^{\xi\eta} = 0, \qquad g_{\xi\eta} = 0, \qquad g^{\eta\eta} = g^{\xi\xi} = \frac{1}{g_{\eta\eta}} = \frac{1}{g_{\xi\xi}}. \tag{3.10.5}$$

Consequently, the matrices of covariant and contravariant metric coefficients are proportional to the identity tensor, $\mathbf{I}$.

3.10.2 Laplacian

The Laplacian of a scalar field, $f(x, y)$, given in its general form in (3.5.30),

$$\nabla^2 f = g^{\xi\xi}\,\frac{\partial^2 f}{\partial \xi^2} + 2\,g^{\xi\eta}\,\frac{\partial^2 f}{\partial \xi \partial \eta} + g^{\eta\eta}\,\frac{\partial^2 f}{\partial \eta^2} + v^{\xi}\frac{\partial f}{\partial \xi} + v^{\eta}\frac{\partial f}{\partial \eta}, \tag{3.10.6}$$

where

$$v^{\xi} \equiv \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\Big(g^{i\xi}\,\sqrt{g}\Big), \qquad v^{\eta} \equiv \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\Big(g^{i\eta}\,\sqrt{g}\Big), \tag{3.10.7}$$

simplifies to

$$\nabla^2 f = \frac{1}{g_{\xi\xi}}\,\widehat{\nabla}^2 f, \tag{3.10.8}$$

where a caret (hat) indicates differentiation with respect to (ξ, η),

$$\widehat{\nabla}^2 f = \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}. \tag{3.10.9}$$

We see that only one component of the metric tensor arises on the right-hand side of (3.10.8). The Poisson equation, $\nabla^2 f + s = 0$ takes the form

$$\widehat{\nabla}^2 f = g_{\xi\xi}\, s, \tag{3.10.10}$$

where s is a distributed source. The Laplace and Poisson equations can be solved using the finite-difference methods discussed previously in this chapter for a channel-like or quadrilateral domain.

3.10.3 Implementation

In the practical application of the method, we introduce a complex function that maps a square or some other simple shape in the ζ plane to the solution domain in the z plane. As an example, the shape of a square mapped by the function

$$F(\zeta) = \frac{1}{\sqrt{2}}\,(1 + \mathrm{i} - 0.9\,\zeta^2)\,(1 - 0.8\,\zeta^4)\,\zeta \tag{3.10.11}$$

is shown in Figure 3.10.1($a,\ b$). Both grids depicted in this figure are orthogonal; the grid in the $\xi\eta$ plane is Cartesian, and the grid in the xy plane is curvilinear. The distributions of the determinant of the matrix of covariant coefficients, g, and contravariant metric coefficient, $g^{\xi\xi}$, are shown in Figure 3.10.1($c,\ d$).

A finite-difference method for solving the Poisson equation with Dirichlet boundary conditions is implemented in a code located in directory Map of Tunlib. For simplicity, the covariant base vectors are computed by central difference approximations. The solution for the homogeneous Dirichlet boundary condition and a uniform source is shown in Figure 3.10.2.

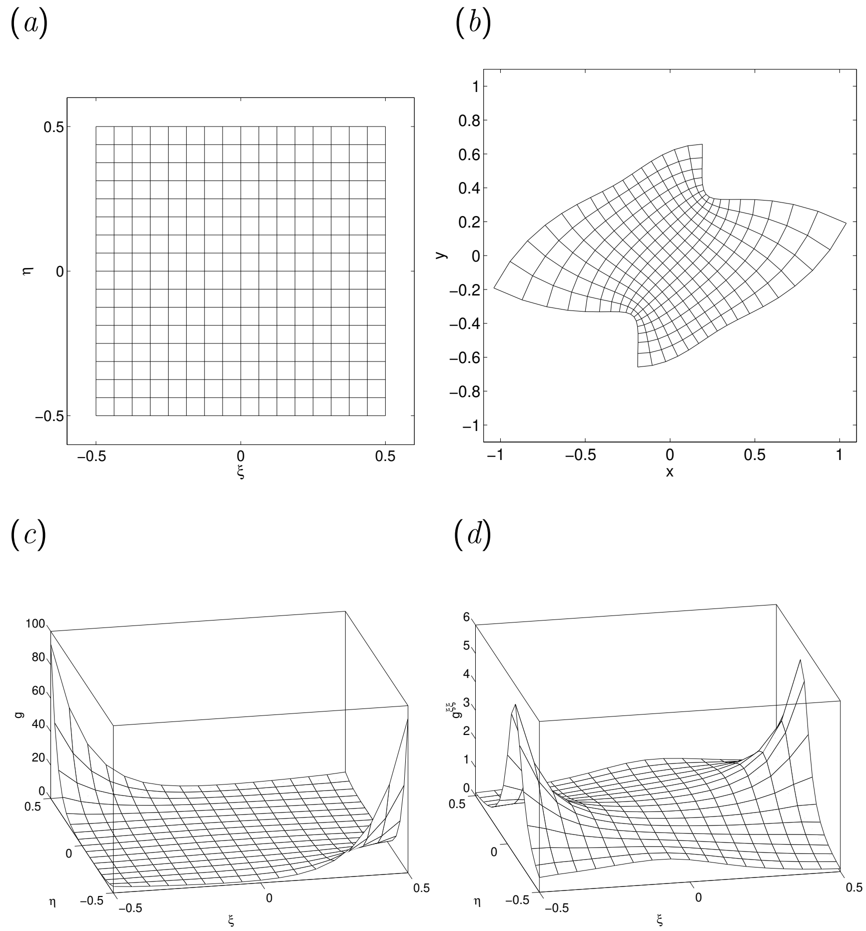

FIGURE 3.10.1 (a, b) Curvilinear coordinates generated by a conformal mapping the unit square centered at the origin of the $\xi\eta$ plane to some shape in the xy plane. (c) Distribution of the determinant of the matrix of covariant coefficients, g. (c, d) Distribution of the contravariant metric coefficient, $g^{\xi\xi}$.

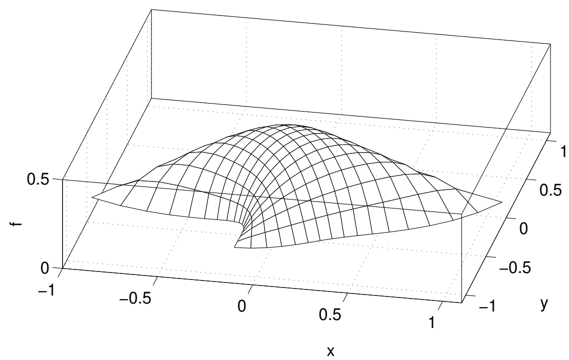

FIGURE 3.10.2 Solution of Poisson's equation by a finite-difference method in a domain generated by conformal mapping.

Exercise

3.10.1 Duplicate Figure 3.10.1 for a mapping function of your choice.

3.11 Elliptic coordinates

For convenience, we denote the contravariant coordinates by $x^1 = \xi$ and $x^2 = \eta$. Elliptic coordinates, (ξ, η), are orthogonal curvilinear coordinates defined by the conformal mapping function

$$x + \mathrm{i}\, y = A \sinh(\xi + \mathrm{i}\, \eta), \qquad (3.11.1)$$

where i is the imaginary unit, $\mathrm{i}^2 = -1$, and A is a real constant. Resolving the mapping function into its real and imaginary parts, we obtain the equations

$$x = A \sinh\xi \cos\eta, \qquad y = A \cosh\xi \sin\eta. \qquad (3.11.2)$$

Fixing the value of ξ provides us with an equation for an ellipse, as shown in Figure 3.11.1. As ξ tends to infinity, the contour lines of constant ξ tend to become concentric circles.

Consider a particular value of $\xi = \xi_0$, and set

$$a = A \sinh\xi_0, \qquad b = A \cosh\xi_0, \qquad (3.11.3)$$

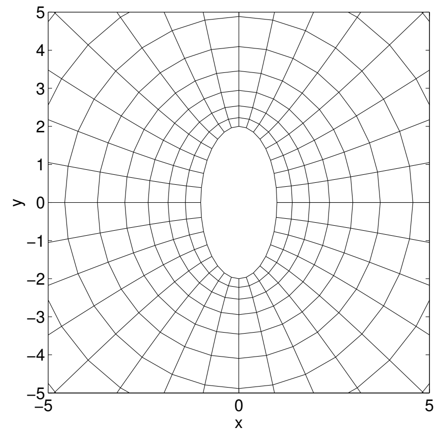

FIGURE 3.11.1 Illustration of grid lines based on elliptic coordinates in the xy plane.

where a and b are the ellipse semi-axes along the x and y axes with $b \geq a$. Solving for A and ξ_0, we find that

$$\tanh \xi_0 = \frac{a}{b}, \qquad A = \frac{a}{\sinh \xi_0}. \tag{3.11.4}$$

These equations can be used to compute ξ_0 and A from specified values of a and b.

3.11.1 Grid generation

The following Matlab code named *elliptic_grid*, located in directory ELLIPTIC of TUNLIB, generates the grid shown in Figure 3.11.1 outside an ellipse with aspect ratio $b/a = 2$:

```
%---
% parameters
%---

a = 1;  % x semi-axis of innermost ellipse
b = 2;  % y semi-axis of innermost ellipse

%---
```

```
% compute xi_0 and A
%---

xi0 = atanh(a/b);
snhxi0 = sinh(xi0); cshxi0 = cosh(xi0);
A = a/snhxi0;

ximax = log(32.0*a/A);  % arbitrary

Np = 2;
Nxi = Np*8;
Net = Np*16;

%---
% grid
%---

 Dxi = (ximax-xi0)/Nxi;

 for i=1:Nxi+1
  xi(i) = xi0+(i-1.0)*Dxi;
  snhxi(i) = sinh(xi(i));
  cshxi(i) = cosh(xi(i));
 end

 Deta = 2.0*pi/Net;

 for j=1:Net+1
  eta(j) = (j-1.0)*Deta;
  cseta(j) = cos(eta(j));
  sneta(j) = sin(eta(j));
 end

%---
% grid points
%---

 for i=1:Nxi+1
  for j=1:Net+1
```

```
   x(i,j) = A*snhxi(i)*cseta(j);
   y(i,j) = A*cshxi(i)*sneta(j);
   J(i,j) = A*sqrt(cshxi(i)^2-sneta(j)^2);
  end
 end

%---
% polar angle around the origin (theta)
% (different than eta)
%---

 xcnt = 0.0; ycnt = 0.0;

 for j=1:Net
  rr = sqrt((x(1,j)-xcnt)^2+(y(1,j)-ycnt)^2);
  theta(j) = acos((x(1,j)-xcnt)/rr);
  if(y(1,j)<0.0)
   theta(j) = 2*pi-theta(j);
  end
 end

 theta(Net+1) = 2*pi+theta(1);

%---
% plot
%---

 for j=1:Net+1
  plot(x(:,j),y(:,j),'r')
 end

 for i=1:Nxi+1
  plot(x(i,:),y(i,:),'k')
 end
```

3.11.2 Base vectors and components of the metric tensor

The covariant base vectors are given by

$$\mathbf{g}_\xi = \frac{\partial \mathbf{x}}{\partial \xi} = A\,\big(\cosh\xi\cos\eta\,\mathbf{e}_x + \sinh\xi\sin\eta\,\mathbf{e}_y\big) \tag{3.11.5}$$

and

$$\mathbf{g}_\eta = \frac{\partial \mathbf{x}}{\partial \eta} = A\,\big(-\sinh\xi\sin\eta\,\mathbf{e}_x + \cosh\xi\cos\eta\,\mathbf{e}_y\big). \tag{3.11.6}$$

The only non-zero covariant components of the metric tensor are the diagonal components

$$g_{\xi\xi} = g_{\eta\eta} = A^2\,(\cosh^2\xi\cos^2\eta + \sinh^2\xi\sin^2\eta\,). \tag{3.11.7}$$

Simplifying, we obtain

$$g_{\xi\xi} = g_{\eta\eta} = A^2\,(\cosh^2\xi - \sin^2\eta\,). \tag{3.11.8}$$

In the Matlab code listed previously in this section, $J = \sqrt{g_{\xi\xi}}$.

Since the coordinates are orthogonal, the contravariant base vectors are aligned with the covariant base vectors. We find that

$$\mathbf{g}^\xi = \frac{1}{g_{\xi\xi}}\,\mathbf{g}_\xi, \qquad \mathbf{g}^\eta = \frac{1}{g_{\eta\eta}}\,\mathbf{g}_\eta, \tag{3.11.9}$$

and thus $g^{\xi\xi} = g^{\eta\eta} = 1/g_{\xi\xi} = 1/g_{\eta\eta}$.

3.11.3 Poisson equation

The Poisson equation, $\nabla^2 f + s = 0$, takes the form

$$\widehat{\nabla}^2 f + g_{\xi\xi}\,s = 0, \tag{3.11.10}$$

where a caret (hat) indicates differentiation with respect to the elliptic coordinates (ξ,η), which can be expanded as

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} + A^2\,(\cosh^2\xi - \sin^2\eta\,)\,s(\xi,\eta) = 0, \tag{3.11.11}$$

where s is a specified source term. The solution domain for an annular region confined between two ellipses is a rectangle confined between

$\xi_0 \le u \le \xi_1$ and $0 \le \eta \le 2\pi$. A periodicity condition is imposed with respect to η.

3.11.4 Finite-difference method

The solution of the Poisson equation can be found by a finite-difference method. The point-Gauss–Seidel (PGS) method is an iterative method according to the following scheme:

$$f_{i,j}^{n+1} = \frac{1}{2\,(1+\beta)} \Big(\, f_{i+1,j}^n + f_{i-1,j}^n + \beta\,(f_{i,j+1}^n + f_{i,j-1}^n) + A^2\,(\cosh^2\xi - \sin^2\eta\,)s_{i,j}\,\Big), \qquad (3.11.12)$$

where the indices i and j parametrize grid points, n is an iteration number and $\beta = (\Delta\xi/\Delta\eta)^2$.

The numerical method with the Dirichlet boundary condition at the inner and outer cylinder is implemented in the following Matlab code named *elliptic_DD*, located in directory ELLIPTIC of TUNLIB:

```
 for j=1:Net+1
  for i=1:Nxi+1
   s(i,j) = 1.0;  % source
  end
 end

 for j=1:Net+1
  for i=1:Nxi+1
   f(i,j) = 0.0;  % initialize
  end
 end

 for j=1:Net+1
  f(1,    j) = 0.0;  % inner boundary condition
  f(Nxi+1,j) = 0.5;  % outer boundary condition
 end

 beta = (Dxi/Det)^2;

%---
```

```
% iterations
%---

 tolerance = 0.0001;

 for iter = 1:200

  for i=2:Nxi  % periodicity condition
    f(i,1)     = f(i,Net+1);
    f(i,Net+2) = f(i,2);
  end

 corrmax = 0.0;
 fc1 = 1/(2.0*(1+beta));
 Dxis = Dxi^2;

 for j=2:Net+1
  for i=2:Nxi

   fold = f(i,j);

   f(i,j) = fc1*( f(i+1,j)+f(i-1,j) ...
          + beta*(f(i,j+1)+f(i,j-1)) ...
          + Dxis*J(i,j)^2*s(i,j) );

    corr = abs(fold-f(i,j));
    if(corr>corrmax) corrmax=corr; end
   end
  end

 if(corrmax<tolerance) break; end

 end % of iterations

%---
% plot
%---

 figure(2)
```

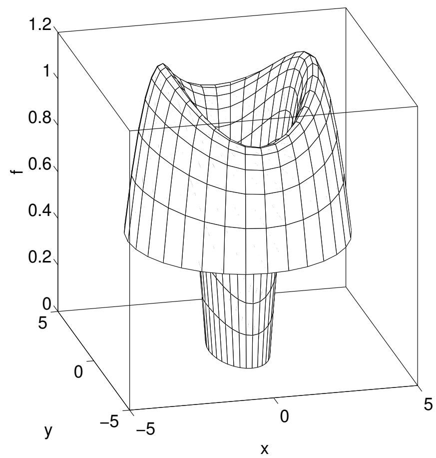

FIGURE 3.11.2 Solution of the Poisson equation with a uniform source computed in elliptic coordinates.

```
surf(x,y,f(1:Nxi+1,1:Net+1))
```

Running the code generates the mushroom-like solution shown in Figure 3.11.2.

3.11.5 Alternative elliptic coordinates

An alternative set of elliptic coordinates, (ξ, η), is defined by the conformal mapping function

$$x + \mathrm{i}\, y = A\, \cosh(\xi + \mathrm{i}\, \eta), \tag{3.11.13}$$

where i is the imaginary unit, $\mathrm{i}^2 = -1$, and A is a real constant. Resolving the mapping function into its real and imaginary parts, we obtain the equations

$$x = A\, \cosh \xi \cos \eta, \qquad y = A\, \sinh \xi \sin \eta. \tag{3.11.14}$$

Consider a particular value of $\xi = \xi_0$, and set

$$a = A\, \cosh \xi_0, \qquad b = A\, \sinh \xi_0, \tag{3.11.15}$$

where a and b are the ellipse semi-axes along the x and y axes with $a \geq b$. Solving for A and ξ_0, we find that

$$\tanh \xi_0 = \frac{b}{a}, \qquad A = \frac{b}{\sinh \xi_0}. \tag{3.11.16}$$

These equations can be used to compute ξ_0 and A from specified semi-axes, a and b.

Exercise

3.11.1 Confirm that $g_{\xi\xi}\,(\mathbf{g}_\xi \otimes \mathbf{g}_\xi + \mathbf{g}_\eta \otimes \mathbf{g}_\eta) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Chapter 4

Non-Cartesian coordinates

The basic concepts, derivations, and equations discussed in Chapter 3 for non-Cartesian, rectilinear or curvilinear coordinates in two dimensions can be extended in a straightforward fashion to three or higher dimensions. In this chapter, we introduce further basic notions, discuss vector and tensor components, study coordinate transformations, and derive expressions for covariant derivatives of vector and tensor components.

In Chapter 5, we will derive expressions for the gradient and other differential operators on vector and tensor fields following a procedure that circumvents a great deal of algebraic manipulations, and then will proceed to study applications.

4.1 Basic framework

A system of nonorthogonal coordinates in three dimensions is shown in Figure 4.1.1. Contravariant lines of constant x^i are drawn as solid lines and covariant lines of constant x_i are drawn as broken lines, where subscripts and superscripts range over $1, 2, 3$.

The covariant base vectors are defined as

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial x^i}, \tag{4.1.1}$$

where $\mathbf{x} = (x, y, z)$ is position in three-dimensional space. The differential displacement is given by

$$\mathrm{d}\mathbf{x} = \mathbf{g}_i \, \mathrm{d}x^i. \tag{4.1.2}$$

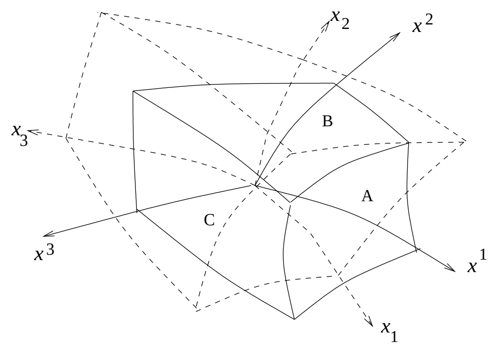

FIGURE 4.1.1 Illustration of three-dimensional nonorthogonal curvilinear coordinates. Contravariant coordinate lines, (x^1, x^2, x^3), are drawn with solid lines and covariant coordinate lines, (x_1, x_2, x_3), are drawn with broken lines.

The contravariant base vectors are defined uniquely by the biorthonormality condition

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_{ij}, \tag{4.1.3}$$

where δ_{ij} is Kronecker's delta. Further properties of the contravariant base vectors are discussed in Section 4.2

4.1.1 Matrices of base vectors

The covariant base vectors can be arranged at the *columns* of a matrix. In three dimensions, this matrix takes the form

$$\mathbf{F} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \partial x/\partial x^1 & \partial x/\partial x^2 & \partial x/\partial x^3 \\ \partial y/\partial x^1 & \partial y/\partial x^2 & \partial y/\partial x^3 \\ \partial z/\partial x^1 & \partial z/\partial x^2 & \partial z/\partial x^3 \end{bmatrix}. \tag{4.1.4}$$

In continuum mechanics, the matrix $\mathbf{F}$ is known as the *deformation gradient*.

The contravariant base vectors, $\mathbf{g}^i$, can be arranged at the *columns* of another matrix,

$$\mathbf{\Phi} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{g}^1 & \mathbf{g}^2 & \mathbf{g}^3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \partial x^1/\partial x & \partial x^2/\partial x & \partial x^3/\partial x \\ \partial x^1/\partial y & \partial x^2/\partial y & \partial x^3/\partial y \\ \partial x^1/\partial z & \partial x^2/\partial z & \partial x^3/\partial z \end{bmatrix}. \tag{4.1.5}$$

Using the rules of matrix multiplication and enforcing the orthogonality property (4.1.3), we find that

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{\Phi} = \mathbf{I}, \qquad \mathbf{\Phi} = \mathbf{F}^{-\mathrm{T}}, \qquad \mathbf{F} = \mathbf{\Phi}^{-\mathrm{T}}, \tag{4.1.6}$$

where the superscript $-\mathrm{T}$ denotes the inverse of the matrix transpose, which is equal to the transpose of the matrix inverse.

4.1.2 Jacobian metric

The Jacobian metric associated with the covariant base vectors, $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$, is the volume of a parallelepiped defined by these vectors, given by

$$\mathcal{J} \equiv \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) = \det(\mathbf{F}) = \frac{1}{\det(\mathbf{\Phi})}. \tag{4.1.7}$$

The physical volume corresponding to a small parallelepiped whose sides are parallel to $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$, is

$$\mathrm{d}V(\mathbf{x}) = \mathcal{J}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3. \tag{4.1.8}$$

This relation can be integrated over a region in contravariant coordinate space to generate the volume of the corresponding region in physical space.

4.1.3 Contravariant from covariant base vectors

Given a set of covariant base vectors in three dimensions, $\mathbf{g}_i$ for $i = 1, 2, 3$, the contravariant base vectors, $\mathbf{g}^i$, can be constructed using the formula

$$\mathbf{g}^i = \tfrac{1}{2}\, \frac{1}{\mathcal{J}}\, \epsilon_{ijk}\, \mathbf{g}_j \times \mathbf{g}_k, \tag{4.1.9}$$

where ϵ_{ijk} is the Levi–Civita symbol and summation is implied over the repeated indices j and k. Explicitly, the contravariant base vectors are given by

$$\mathbf{g}^1 = \frac{1}{\mathcal{J}}\, \mathbf{g}_2 \times \mathbf{g}_3, \quad \mathbf{g}^2 = \frac{1}{\mathcal{J}}\, \mathbf{g}_3 \times \mathbf{g}_1, \quad \mathbf{g}^3 = \frac{1}{\mathcal{J}}\, \mathbf{g}_1 \times \mathbf{g}_2. \tag{4.1.10}$$

These formulas can be expressed collectively as

$$\mathbf{g}_i \times \mathbf{g}_j = \mathcal{J}\,\epsilon_{ijk}\,\mathbf{g}^k, \tag{4.1.11}$$

which confirms that $\mathbf{g}^k$ is perpendicular to $\mathbf{g}_i$ and $\mathbf{g}_j$, for $k \neq i, j$. Taking the inner product of this equation with an arbitrary covariant base vector, $\mathbf{g}_m$, we find that

$$(\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_m = \mathcal{J}\,\epsilon_{ijm}. \tag{4.1.12}$$

For $j = 1$, $j = 2$, and $m = 3$, we recover (4.1.7).

4.1.4 Covariant from contravariant base vectors

In three dimensions, the covariant base vectors can be computed from the contravariant base vectors using the expressions

$$\mathbf{g}_i = \tfrac{1}{2}\,\mathcal{J}\,\epsilon_{ijk}\,\mathbf{g}^j \times \mathbf{g}^k. \tag{4.1.13}$$

Explicitly,

$$\mathbf{g}_1 = \mathcal{J}\,\mathbf{g}^2 \times \mathbf{g}^3, \quad \mathbf{g}_2 = \mathcal{J}\,\mathbf{g}^3 \times \mathbf{g}^1, \quad \mathbf{g}_3 = \mathcal{J}\,\mathbf{g}^1 \times \mathbf{g}^2. \tag{4.1.14}$$

These formulas can be compiled into the form

$$\mathbf{g}^i \times \mathbf{g}^j = \frac{1}{\mathcal{J}}\,\epsilon_{ijk}\,\mathbf{g}_k, \tag{4.1.15}$$

which confirms that $\mathbf{g}_k$ is perpendicular to $\mathbf{g}^i$ and $\mathbf{g}^j$ for $k \neq i, j$.

Exercise

4.1.1 Combining (4.1.12) and (4.1.9), we find that

$$\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \tfrac{1}{2}\,\frac{1}{\mathcal{J}}\,\epsilon_{ijk}\,(\mathbf{g}_j \times \mathbf{g}_k) \cdot (\mathbf{g}_m \mathrm{d}x^m) = \tfrac{1}{2}\,\epsilon_{ijk}\epsilon_{jkm}\,\mathrm{d}x^m. \tag{4.1.16}$$

Use the properties of the Levi–Civita symbol to obtain $\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \mathrm{d}x^i$, as shown in (4.2.3).

4.2 Contravariant base vectors

Combining the expression for the contravariant base vectors given in (4.1.9), with the expression for the differential displacement given in (4.1.2), we find that

$$\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \tfrac{1}{2}\,\frac{1}{\mathcal{J}}\,\epsilon_{ijk}\,(\mathbf{g}_j \times \mathbf{g}_k)\cdot(\mathbf{g}_m \mathrm{d}x^m). \tag{4.2.1}$$

Now using (4.1.12) we obtain

$$\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \tfrac{1}{2}\,\epsilon_{ijk}\,\epsilon_{jkm}\,\mathrm{d}x^m. \tag{4.2.2}$$

Recalling the properties of the Levi–Civita symbol, we obtain

$$\mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = \mathrm{d}x^i. \tag{4.2.3}$$

Integrating this equation between two arbitrary points, A and B, we find that

$$\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = (x^i)_{\mathrm{B}} - (x^i)_{\mathrm{A}}. \tag{4.2.4}$$

Consequently,

$$\oint \mathbf{g}^i \cdot \mathrm{d}\mathbf{x} = 0, \tag{4.2.5}$$

where the integration is performed along an arbitrary closed contour in space.

4.2.1 Gradient of contravariant coordinates

The contravariant coordinates, x^i, can be regarded as functions of position, $\mathbf{x}$. We will show that the contravariant base vectors are the gradients of the contravariant coordinates,

$$\mathbf{g}^i = \boldsymbol{\nabla} x^i, \tag{4.2.6}$$

where $\boldsymbol{\nabla}$ is the gradient operator. Physically, $\mathbf{g}^i$ points in the direction of maximum increase of x^i with respect to arc length. To compute the gradients on the right-hand side of (4.2.6), the functions $x^i(\mathbf{x})$ must be available, which is the case only for simple configurations.

Since the curl of the gradient of any function is identically zero, the contravariant base vector fields are irrotational,

$$\nabla \times \mathbf{g}^i = \mathbf{0}. \tag{4.2.7}$$

Using the Stokes circulation theorem, we find that the circulation of each contravariant base vector along any arbitrary closed loop in space is zero, as shown in (4.2.5).

4.2.2 *Inverses of vector functions*

To demonstrate (4.2.6), we consider a vector field, $\mathbf{f}$, that is a function of another vector field, $\mathbf{q}$, and its inverse,

$$\mathbf{f} = \mathbf{F}(\mathbf{q}), \qquad \mathbf{q} = \mathbf{Q}(\mathbf{f}). \tag{4.2.8}$$

In the present context, the vectors $\mathbf{f}$ and $\mathbf{q}$ are interpreted as one-dimensional arrays. We may write

$$\mathrm{d}\mathbf{f} = \mathrm{d}\mathbf{q} \cdot \nabla_q \mathbf{F}(\mathbf{q}), \qquad \mathrm{d}\mathbf{q} = \mathrm{d}\mathbf{f} \cdot \nabla_f \mathbf{Q}(\mathbf{f}), \tag{4.2.9}$$

where the gradient ∇_q operates with respect to $\mathbf{q}$ and the gradient ∇_f operates with respect to $\mathbf{f}$. Combining these equations, we find that

$$\nabla \mathbf{F}_q(\mathbf{q}) \cdot \nabla_f \mathbf{Q}(\mathbf{f}) = \mathbf{I}, \tag{4.2.10}$$

where $\mathbf{I}$ is the identity matrix. In index notation,

$$\frac{\partial F_\gamma}{\partial q_\alpha} \frac{\partial Q_\beta}{\partial f_\gamma} = \delta_{\alpha\beta}, \tag{4.2.11}$$

where $\delta_{\beta\gamma}$ is Kronecker's delta and Greek indices indicate array entries.

The contravariant coordinates, x^i, can be regarded as functions of position, $\mathbf{x}$, and vice versa,

$$x^i = F_i(\mathbf{x}), \qquad x_\alpha = Q_\alpha(x^1, x^2, x^3). \tag{4.2.12}$$

Setting in (4.2.11)

$$F_\gamma = x_\gamma, \qquad f_\gamma = x_\gamma, \qquad q_\alpha = x^i, \qquad Q_\beta = x^j, \tag{4.2.13}$$

we obtain

$$\frac{\partial \mathbf{x}}{\partial x^i} \cdot \boldsymbol{\nabla} x^j = \mathbf{g}_i \cdot \boldsymbol{\nabla} x^j = \delta_{ij}, \tag{4.2.14}$$

which confirms that the contravariant base vectors are given by (4.2.6).

Exercise

4.2.1 Discuss Stokes's circulation theorem in three-dimensional space.

4.3 Metric coefficients

The covariant and contravariant metric coefficients, defined as

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j, \qquad g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j, \tag{4.3.1}$$

can be collected into two matrices denoted by

$$\mathbf{g} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \boldsymbol{\Phi}^{-1} \cdot \boldsymbol{\Phi}^{-\mathrm{T}} \tag{4.3.2}$$

and

$$\boldsymbol{\gamma} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \boldsymbol{\Phi}^{\mathrm{T}} \cdot \boldsymbol{\Phi} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-\mathrm{T}}. \tag{4.3.3}$$

Multiplying these expressions, we find that

$$\mathbf{g} \cdot \boldsymbol{\gamma} = \mathbf{I}, \tag{4.3.4}$$

which shows that $\boldsymbol{\gamma}$ is the inverse of $\mathbf{g}$, and *vice versa*.

4.3.1 Jacobian metric

Since the determinant of a matrix is equal to the determinant of the transpose, we find from (4.3.2) or (4.3.3) that $g = \mathcal{J}^2$ or

$$\mathcal{J} = \sqrt{g}, \tag{4.3.5}$$

where

$$g \equiv \det(\mathbf{g}) = \frac{1}{\det(\boldsymbol{\gamma})} \tag{4.3.6}$$

and $\mathcal{J} = \det(\mathbf{F})$.

4.3.2 Fundamental form of space

Using the definitions of the covariant metric coefficients, we find that the square of the magnitude of the differential displacement is given by

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = (\mathbf{g}_i \, \mathrm{d}x^i) \cdot (\mathbf{g}_j \, \mathrm{d}x^j), \tag{4.3.7}$$

where summation is implied over the repeated indices, i and j. Distributing the multiplications and invoking the definition of the covariant metric coefficients, we derive the fundamental form of space,

$$\mathrm{d}s^2 \equiv \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = g_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j, \tag{4.3.8}$$

where $\mathrm{d}s^2$ is the square of differential distance. We see that the differential distance is defined with respect to the contravariant coordinates and covariant metric coefficients.

4.3.3 Base vector conversion

Rearranging (4.3.2)), we find that

$$\mathbf{F} = \mathbf{g} \cdot \mathbf{F}^{-1} = \mathbf{g} \cdot \boldsymbol{\Phi}^{\mathrm{T}}, \qquad \boldsymbol{\Phi} = \boldsymbol{\gamma} \cdot \boldsymbol{\Phi}^{-1} = \boldsymbol{\gamma} \cdot \mathbf{F}^{\mathrm{T}}, \tag{4.3.9}$$

which shows that

$$\mathbf{g}_i = g_{ij} \, \mathbf{g}^j, \qquad \mathbf{g}^i = g^{ij} \, \mathbf{g}_j. \tag{4.3.10}$$

These expressions provide us with rules for raising and lowering indices.

4.3.4 Coordinate surface metrics

The differential surface area of the face labeled A in Figure 4.1.1 is given by

$$\mathrm{d}S_{\mathrm{A}} = |\mathbf{g}_2 \times \mathbf{g}_3| \, \mathrm{d}x^2 \, \mathrm{d}x^3 = \mathcal{J} \, |\mathbf{g}^1| \, \mathrm{d}x^2 \, \mathrm{d}x^3. \tag{4.3.11}$$

Setting $|\mathbf{g}^1| = \sqrt{g^{11}}$, we obtain

$$\mathrm{d}S_\mathrm{A} = \mathcal{J}\,\sqrt{g^{11}}\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{4.3.12}$$

The unit vector normal to this surface is

$$\mathbf{n}_\mathrm{A} = \frac{\mathbf{g}^1}{\sqrt{g^{11}}}. \tag{4.3.13}$$

Combining the last two equations, we obtain

$$\mathbf{n}_\mathrm{A}\,\mathrm{d}S_\mathrm{A} = \mathcal{J}\,\mathbf{g}^1\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{4.3.14}$$

This expression can be used to compute the flux of a vector field across surface A, as discussed in Section 4.3.

4.3.5 Summary

An assortment of relations are given in Table 4.3.1. To prove the sixth relation, we note that

$$\frac{\partial g}{\partial g_{ij}} = g^\mathrm{c}_{ij} = g\,g^{-1}_{ji} = g\,g^{ji}, \tag{4.3.15}$$

where g^c_{ij} is the cofactor of the element g_{ij} in the matrix $\mathbf{g}$. Substituting $g = \mathcal{J}^2$ yields the aforementioned relation. The seventh relation in Table 4.3.1 arises by applying the chain rule and invoking the sixth relation.

Exercise

4.3.1 Derive the seventh relation in Table 4.3.1.

4.4 Covariant coordinates

A family of non-intersecting lines can be drawn that are tangential to the contravariant base vectors, $\mathbf{g}^i$ for $i = 1, 2, 3$. The position along a line in the ith family can be parametrized by a coordinate x^i. The triplet (x_1, x_2, x_3), constitutes *covariant* curvilinear coordinates.

1	$\mathbf{I} = g_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i = g^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j$
2	$\boldsymbol{\gamma} = \mathbf{g}^{-1}$
3	$\mathcal{J} = \sqrt{\det(\mathbf{g})} = \frac{1}{\sqrt{\det(\boldsymbol{\gamma})}}$
4	$\mathbf{g}_i = g_{ij}\,\mathbf{g}^j$
5	$\mathbf{g}^i = g^{ij}\,\mathbf{g}_j$
6	$\dfrac{\partial \mathcal{J}}{\partial g_{ij}} = \frac{1}{2}\,\mathcal{J}\,g^{ij}$
7	$\dfrac{\partial \mathcal{J}}{\partial x^k} = \frac{1}{2}\,\mathcal{J}\,g^{ij}\,\dfrac{\partial g_{ij}}{\partial x^k}$

TABLE 4.3.1 An assortment of relations pertaining to general, orthogonal or nonorthogonal curvilinear coordinates.

4.4.1 Position as a function of covariant coordinates

The position in three-dimensional space can be regarded as a function of covariant coordinates,

$$\mathbf{x}(x_1, x_2, x_3). \tag{4.4.1}$$

An infinitesimal displacement vector can be expressed as

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial \mathbf{x}}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial \mathbf{x}}{\partial x_3}\,\mathrm{d}x_3, \tag{4.4.2}$$

where $\mathrm{d}x_1$, $\mathrm{d}x_2$, and $\mathrm{d}x_2$ are differential increments regarded as covariant components of the differential displacement, $\mathrm{d}\mathbf{x}$.

Since $\partial \mathbf{x}/\partial x_i$ is parallel to $\mathbf{g}^1$, we can introduce three appropriate functions, $\alpha_1(x_1, x_2, x_3)$, $\alpha_2(x_1, x_2, x_3)$, and $\alpha_3(x_1, x_2, x_3)$, and write

$$\frac{\partial \mathbf{x}}{\partial x_1} = \frac{1}{\alpha_1}\,\mathbf{g}^1, \qquad \frac{\partial \mathbf{x}}{\partial x_2} = \frac{1}{\alpha_2}\,\mathbf{g}^2, \qquad \frac{\partial \mathbf{x}}{\partial x_3} = \frac{1}{\alpha_2}\,\mathbf{g}^3. \tag{4.4.3}$$

Consequently,

$$\mathrm{d}\mathbf{x} = \frac{1}{\alpha_1}\,\mathbf{g}^1 \mathrm{d}x_1 + \frac{1}{\alpha_2}\,\mathbf{g}^2 \mathrm{d}x_2 + \frac{1}{\alpha_3}\,\mathbf{g}^3 \mathrm{d}x_3, \tag{4.4.4}$$

as discussed in Section 4.2.

The covariant coordinates can be regarded as functions of the contravariant coordinates. Using the chain rule, we write

$$\mathbf{g}_i \equiv \frac{\partial \mathbf{x}}{\partial x^i} = \frac{\partial \mathbf{x}}{\partial x_j}\frac{\partial x_j}{\partial x^i}, \tag{4.4.5}$$

yielding

$$\mathbf{g}_i = \frac{1}{\alpha_1}\,\mathbf{g}^1 \frac{\partial x_1}{\partial x^i} + \frac{1}{\alpha_2}\,\mathbf{g}^2 \frac{\partial x_2}{\partial x^i} + \frac{1}{\alpha_3}\,\mathbf{g}^3 \frac{\partial x_3}{\partial x^i} \tag{4.4.6}$$

for $i = 1, 2, 3$. Comparing this expression with the rule for lowering an index, $\mathbf{g}_i = g_{ij}\mathbf{g}^j$, we obtain the relations

$$g_{i1} = \frac{1}{\alpha_1}\,\frac{\partial x_1}{\partial x^i}, \qquad g_{i2} = \frac{1}{\alpha_2}\,\frac{\partial x_2}{\partial x^i}, \qquad g_{i3} = \frac{1}{\alpha_3}\,\frac{\partial x_3}{\partial x^i}, \tag{4.4.7}$$

which can be integrated to provide us with the covariant coordinate field, x_1, x_2, and x_3.

4.4.2 Compatibility conditions

Equations (4.4.7) require that the functions α_1, α_2, and α_3 satisfy the compatibility conditions

$$\frac{\partial(\alpha_1\, g_{i1})}{\partial x^k} = \frac{\partial(\alpha_1\, g_{k1})}{\partial x^i}, \qquad \frac{\partial(\alpha_2\, g_{i2})}{\partial x^k} = \frac{\partial(\alpha_2\, g_{k2})}{\partial x^i},$$
$$\frac{\partial(\alpha_3\, g_{i3})}{\partial x^k} = \frac{\partial(\alpha_3\, g_{k3})}{\partial x^i} \tag{4.4.8}$$

for any pair, i, k. These compatibility conditions are not necessarily satisfied when $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = 1$. Once α_1, α_2, and α_3 are specified in agreement with the compatibility conditions, the covariant coordinates can be deduced, as discussed in Section 4.2.

4.4.3 Orthogonal coordinates

The compatibility conditions for orthogonal coordinates require that

$$\begin{aligned} &\frac{\partial(\alpha_1\, g_{11})}{\partial x^2} = 0, \qquad &&\frac{\partial(\alpha_1\, g_{11})}{\partial x^3} = 0, \\ &\frac{\partial(\alpha_2\, g_{22})}{\partial x^1} = 0, \qquad &&\frac{\partial(\alpha_2\, g_{22})}{\partial x^3} = 0, \\ &\frac{\partial(\alpha_2\, g_{33})}{\partial x^1} = 0, \qquad &&\frac{\partial(\alpha_2\, g_{33})}{\partial x^2} = 0. \end{aligned} \tag{4.4.9}$$

Integrating these equations, we obtain

$$\alpha_1 = \frac{1}{g_{11}}\,\mathcal{A}(x^1), \qquad \alpha_2 = \frac{1}{g_{22}}\,\mathcal{B}(x^2), \qquad \alpha_3 = \frac{1}{g_{33}}\,\mathcal{C}(x^3), \tag{4.4.10}$$

where $\mathcal{A}(x^1)$, $\mathcal{B}(x^2)$, and $\mathcal{C}(x^3)$, are arbitrary functions. Integrating equations (4.4.7), we obtain

$$x_1 = \int \mathcal{A}(x^1)\,\mathrm{d}x^1, \qquad x_2 = \int \mathcal{B}(x^2)\,\mathrm{d}x^2,$$
$$x_3 = \int \mathcal{B}(x^3)\,\mathrm{d}x^3. \tag{4.4.11}$$

For $\mathcal{A}(x^1) = 1$ $\mathcal{B}(x^2) = 1$, and $\mathcal{C}(x^2) = 1$, we find that $x_1 = x^1$, $x_2 = x^2$, $x_3 = x^3$, and

$$\frac{\partial \mathbf{x}}{\partial x_1} = g_{11}\,\mathbf{g}^1, \qquad \frac{\partial \mathbf{x}}{\partial x_2} = g_{22}\,\mathbf{g}^2, \qquad \frac{\partial \mathbf{x}}{\partial x_3} = g_{33}\,\mathbf{g}^3. \tag{4.4.12}$$

4.4.4 Construction

More generally, the covariant coordinate field can be constructed using a procedure that is similar to that discussed in Section 3.2.

Exercise

4.4.1 Derive a set of covariant coordinates associated with spherical polar coordinates.

4.5 Vectors

A vector, $\mathbf{v}$, can be expanded in terms of covariant base vectors, $\mathbf{g}_i$, or contravariant base vectors, $\mathbf{g}^i$, in two combinations, as

$$\mathbf{v} = v^i\,\mathbf{g}_i = v_i\,\mathbf{g}^i, \tag{4.5.1}$$

where v^i are the contravariant vector components, v_i are the covariant vector components, and summation is implied over the repeated index, i. Projecting (4.5.1) on a covariant or contravariant base vector, we find that

$$v^i = \mathbf{v} \cdot \mathbf{g}^i, \qquad v_i = \mathbf{v} \cdot \mathbf{g}_i. \tag{4.5.2}$$

To deduce the contravariant vector components from the first equation, the contravariant base vectors, $\mathbf{g}^i$, must be available.

4.5.1 Contravariant from covariant components

Projecting the decomposition (4.5.1) onto $\mathbf{g}_j$ or $\mathbf{g}^j$, where j is a free index, we find that the components are related by

$$v^i = g^{ij} v_j, \qquad v_i = g_{ij} v^j, \tag{4.5.3}$$

where summation is implied over the repeated index, j. These expressions provide us with ways of raising or lowering the indices, thereby deducing the contravariant from the covariant components, and *vice versa*.

4.5.2 Inner vector product

The inner product of two vectors, $\mathbf{u}$ and $\mathbf{v}$, is a scalar,

$$\mathbf{u} \cdot \mathbf{v} \equiv (u^i\,\mathbf{g}_i) \cdot (v_j\,\mathbf{g}^j) = u^i v_j\,\mathbf{g}_i \cdot \mathbf{g}^j = u^i v_j\,\delta_{ij}. \tag{4.5.4}$$

Simplifying, we obtain

$$\mathbf{u} \cdot \mathbf{v} = u^i v_i = v^i u_i \tag{4.5.5}$$

for any pair of contravariant–covariant components.

4.5.3 Vector flux

Using expression (4.3.14), repeated below for convenience,

$$\mathbf{n}_\mathrm{A} \, \mathrm{d}S_\mathrm{A} = \mathcal{J} \, \mathbf{g}^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3, \tag{4.5.6}$$

we find that the flux of vector field, $\mathbf{v}$, across the face labeled A in Figure 4.5.1 is given by

$$\mathbf{v} \cdot \mathbf{n}_\mathrm{A} \, \mathrm{d}S_\mathrm{A} = \mathcal{J} \, \mathbf{v} \cdot \mathbf{g}^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 = \mathcal{J} \, v^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3, \tag{4.5.7}$$

involving the first contravariant component of $\mathbf{v}$. Similar expressions can be written for the faces labeled B and C in Figure 4.5.1.

4.5.4 Divergence of a vector field

Working as in Section 2.4.3, we apply the divergence theorem over a differential coordinate volume element and find that the divergence of a vector field, $\mathbf{v}$, is given by

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{\sqrt{g}} \, \frac{\partial}{\partial x^i} \big(v^i \, \sqrt{g} \big), \tag{4.5.8}$$

where summation is implied over the repeated index, i. The derivatives on the right-hand side can be discretized by standard numerical methods.

4.5.5 Cross product

Working as in Section 3.4.3, we find that the cross product of two vectors, $\mathbf{v}$ and $\mathbf{u}$, is given by

$$\mathbf{v} \times \mathbf{u} = \frac{1}{\mathcal{J}} \, \epsilon_{ijk} \, v_i u_j \, \mathbf{g}_k = \mathcal{J} \, \epsilon_{ijk} \, v^i u^j \, \mathbf{g}^k, \tag{4.5.9}$$

involving corresponding pairs of vector components.

Exercise

4.5.1 Derive expression (4.5.8).

4.6 Biorthogonal v. curvilinear

Biorthogonal bases discussed in Chapter 2 involve two dual sets of base vectors, $\mathbf{b}_i$ and $\mathbf{b}_i$, that satisfy the orthogonality conditions

$$\mathbf{b}_i \cdot \mathbf{b}^j = \omega^{(i)}\, \delta_{ij}, \tag{4.6.1}$$

where $\omega^{(i)} \equiv \mathbf{b}_i \cdot \mathbf{b}^i$, summation is not implied over the repeated index, i, and δ_{ij} is Kronecker's delta representing the identity matrix: $\delta_{ij} = 1$ if $i = j$, or 0 otherwise.

The framework of non-orthogonal coordinates discussed in this chapter can be regarded as a specialization and an extension of the apparatus of biorthogonal bases discussed in Chapter 3. Correspondence of symbols and notions and symbols is summarized in Table 4.6.1. We note, in particular, that the coefficients $\omega^{(i)}$ in general biorthogonal bases are arbitrary, but restricted to be unity in the framework of curvilinear coordinates.

In the framework of curvilinear coordinates, contravariant and covariant are introduced, and base vectors are regarded as functions of position mediated by contravariant and covariant coordinates. Spatial variations of base vectors may then be considered by rates that are quantified by the Christofell symbols.

Exercise

4.6.1 Add to Table 4.6.1 another entry of your choice.

4.7 Tensors

A tensor, $\mathbf{T}$, can be expressed in terms of covariant or contravariant base vectors in four combinations,

$$\begin{aligned}\mathbf{T} = T^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j = T^{i}_{\circ j}\, \mathbf{g}_i \otimes \mathbf{g}^j \\ = T^{\circ j}_{i}\, \mathbf{g}^i \otimes \mathbf{g}_j = T_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j,\end{aligned} \tag{4.7.1}$$

where summation is implied over the repeated indices, i and j; we recall

General biorthogonal	Curvilinear	
$\mathbf{b}_i$	$\mathbf{g}_i = \partial \mathbf{x}/\partial x^i$	covariant base vectors
$\mathbf{b}^i$	$\mathbf{g}^i = \boldsymbol{\nabla} x^i$	contravariant base vectors
$\mathbf{b}_i \cdot \mathbf{b}^i = \omega^{(i)}$	$\mathbf{g}_i \cdot \mathbf{g}^i = 1$	no summation over i.
b_{ij}	g_{ij}	covariant metric coefficients
b^{ij}	g^{ij}	contravariant metric coefficients
$\mathbf{b}$	$\mathbf{g}$	matrix of covariant metric coefficients
$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	matrix of contravariant metric coefficients
$\mathcal{J}_\circ$	$\mathcal{J}$	Jacobian
$\mathcal{J}^\circ$	$\dfrac{1}{\mathcal{J}}$	Jacobian
$\mathbf{b}^i = \dfrac{1}{\omega^{(j)}}\, b^{ji}\, \mathbf{b}_j$	$\mathbf{g}_i = g_{ji}\, \mathbf{g}^j$	Conversion
$\mathbf{b}_i = \dfrac{1}{\omega^{(j)}}\, b_{ji}\, \mathbf{b}^j$	$\mathbf{g}^i = g^{ji}\, \mathbf{g}_j$	Conversion
$v^i = \dfrac{1}{\omega^{(i)}} b^{ij}\, v_j$	$v^i = g^{ij}\, v_j$	Conversion
$v_i = \dfrac{1}{\omega^{(i)}} b_{ij}\, v^j$	$v_i = g_{ij}\, v^j$	Conversion

TABLE 4.6.1 Correspondence of notation and symbols for biorthogonal and curvilinear bases.

that the circular symbol, $\circ$, is a blank space holder. The coefficients T^{ij} are the contravariant tensor components, the coefficients T_{ij} are the covariant components, and the coefficients $T^{i}_{\circ j}$ and $T_i^{\circ j}$ are mixed components.

Projecting these expansions onto covariant or contravariant base vectors, we find that

$$T^{i}_{\circ j} = g^{ik} T_{kj}, \qquad T_{ij} = g_{ik} T^{k}_{\circ j}, \qquad T^{ij} = g^{ik} T_{km} g^{mj}, \tag{4.7.2}$$

and other similar relations for raising or lowering indices, where summation is implied over repeated indices.

4.7.1 Tensor transpose

The transport of a tensor, $\mathbf{T}$, is given by

$$\begin{aligned} \mathbf{V} \equiv \mathbf{T}^{\mathrm{T}} = T^{ij}\, \mathbf{g}_j \otimes \mathbf{g}_i = T^{i}_{\circ j}\, \mathbf{g}^j \otimes \mathbf{g}_i \\ = T_i^{\circ j}\, \mathbf{g}_j \otimes \mathbf{g}^i = T_{ij}\, \mathbf{g}^j \otimes \mathbf{g}^i, \end{aligned} \tag{4.7.3}$$

where summation is implied over the repeated indices, i and j. Note that transposing amounts to changing the order the base vectors in the tensor product. We see that

$$V^{ji} = T^{ij}, \qquad S_j^{\circ i} = T^{i}_{\circ j}, \qquad S^{j}_{\circ i} = T_i^{\circ j}, \qquad S_{ji} = T_{ij}. \tag{4.7.4}$$

In the case of a symmetric tensor, we set $\mathbf{V} = \mathbf{T}$ and obtain

$$T^{ij} = T^{ji}, \qquad T^{i}_{\circ j} = T_j^{\circ i}, \qquad T_{ij} = T_{ji}. \tag{4.7.5}$$

In the case of a skew-symmetric tensor, we set $\mathbf{V} = -\mathbf{T}$ and obtain

$$T^{ij} = -T^{ji}, \qquad T^{i}_{\circ j} = -T_j^{\circ i}, \qquad T_{ij} = -T_{ji}. \tag{4.7.6}$$

Note that, for a skew-symmetric tensor, the diagonal elements of T^{ij} and T_{ij} are zero.

4.7.2 Product of two tensors

The product of two tensors, $\mathbf{T}$ and $\mathbf{S}$, is another tensor given by

$$\mathbf{W} \equiv \mathbf{T} \cdot \mathbf{S} = (T^{i}_{\circ m}\, \mathbf{g}_i \otimes \mathbf{g}^m) \cdot (S^{nj}\, \mathbf{g}_n \otimes \mathbf{g}_j). \tag{4.7.7}$$

We find that

$$\mathbf{W} = T^{i}_{\ \circ m}\, S^{nj}\, (\mathbf{g}_i \otimes \mathbf{g}^m) \cdot (\mathbf{g}_n \otimes \mathbf{g}_j), \tag{4.7.8}$$

and then

$$\mathbf{W} = T^{i}_{\ \circ m}\, S^{nj}\, (\mathbf{g}^m \cdot \mathbf{g}_n)\, \mathbf{g}_i \otimes \mathbf{g}_j, \tag{4.7.9}$$

yielding

$$\mathbf{W} = T^{i}_{\ \circ n} S^{nj}\, \mathbf{g}_i \otimes \mathbf{g}_j. \tag{4.7.10}$$

The pure contravariant components of $\mathbf{W}$ are thus given by

$$w^{ij} = T^{i}_{\ \circ n} S^{nj}. \tag{4.7.11}$$

Working in a similar fashion, we find that

$$\begin{aligned} \mathbf{W} = T_i^{\ \circ n} S_{nj}\, \mathbf{g}^i \otimes \mathbf{g}^j = T_{in} S^{nj}\, \mathbf{g}^i \otimes \mathbf{g}_j \\ = T^{in} S_{nj}\, \mathbf{g}_i \otimes \mathbf{g}^j = T^{i}_{\ \circ n} S^{nj}\, \mathbf{g}_i \otimes \mathbf{g}_j. \end{aligned} \tag{4.7.12}$$

4.7.3 Metric tensor

Using the orthogonality properties of the base vectors, we find that the metric tensor is the identity matrix,

$$\mathbf{G} = g_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i = g^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{I}. \tag{4.7.13}$$

Using the gradient form of the contravariant base vectors shown in (4.2.6), we find that

$$\mathbf{I} = g_{ij}\, \boldsymbol{\nabla} x^i \otimes \boldsymbol{\nabla} x^j. \tag{4.7.14}$$

4.7.4 Tensor inverse

The inverse of a tensor, $\mathbf{T}$, denoted by $\mathbf{S} \equiv \mathbf{T}^{-1}$, satisfies (4.7.12) with $\mathbf{W} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Recalling the representation of the identity matrix shown in (4.7.13), we find that

$$\begin{aligned} T_i^{\ \circ n} S_{nj} = g_{ij}, \qquad T_{in} S^{nj} = \delta_{ij}, \\ T^{in} S_{nj} = \delta_{ij}, \qquad T^{i}_{\ \circ n} S^{nj} = g^{ij}. \end{aligned} \tag{4.7.15}$$

The second and third relations imply that

$$[T_{ij}] = [S^{ij}]^{-1}, \qquad [T^{ij}] = [S_{ij}]^{-1}, \tag{4.7.16}$$

where $[T_{ij}]$ is matrix whose ijth element is T_{ij}. Similar definitions apply for the other three matrices.

4.7.5 Double-dot product

The double-dot product of two tensors, $\mathbf{T}$ and $\mathbf{S}$, is a scalar,

$$\mathbf{T} : \mathbf{S} \equiv \mathrm{trace}(\mathbf{T}^{\mathrm{T}} \cdot \mathbf{S}) = \mathrm{trace}(\mathbf{T} \cdot \mathbf{S}^{\mathrm{T}}) \tag{4.7.17}$$

or

$$\mathbf{T} : \mathbf{S} = T_{ij} S^{ij} = T^{ij} S_{ij}, \tag{4.7.18}$$

where the superscript T denotes the matrix transpose and summation is implied over the repeated indices, i and j.

4.7.6 Finite-volume method

Consider a control volume bounded by three pairs of coordinate surfaces where one coordinate is constant over each surface, as shown in Figure 4.1.1. Applying the divergence theorem in three dimensions for a tensor field, $\mathbf{T}$, over a volume enclosed by three pairs of coordinate surfaces, we obtain

$$\iiint \boldsymbol{\nabla} \cdot \mathbf{T} \, \mathrm{d}V \tag{4.7.19}$$

and then

$$\iiint \boldsymbol{\nabla} \cdot \mathbf{T} \, \mathrm{d}V = \iint \mathbf{n} \cdot \mathbf{T} \, \mathrm{d}S, \tag{4.7.20}$$

where the surface integral is computed over the six faces of the control volume and $\mathbf{n}$ is the outward unit normal vector.

Invoking expression (4.3.14) for the product of the normal vector and surface area, we find that the surface integral over face A is

$$\mathbf{F}_{\mathrm{A}} = \iint_{\mathrm{A}} \mathbf{g}^1 \cdot \mathbf{T} \, \mathcal{J} \, \mathrm{d}x^2 \, \mathrm{d}x^3. \tag{4.7.21}$$

Substituting the pure covariant expansion $\mathbf{T} = T^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j$, where summation is implied over the repeated indices i and j, we obtain

$$\mathbf{F}_\mathrm{A} = \iint_\mathrm{A} T^{ij}\,(\mathbf{g}^1 \cdot \mathbf{g}_i)\,\mathbf{g}_j\,\mathcal{J}\,\mathrm{d}x^2\,\mathrm{d}x^3 \qquad (4.7.22)$$

or

$$\mathbf{F}_\mathrm{A} = \iint_\mathrm{A} T^{1j}\,\mathbf{g}_j\,\mathcal{J}\,\mathrm{d}x^2\,\mathrm{d}x^3. \qquad (4.7.23)$$

For example, in fluid mechanics, $\mathbf{T}$ can be the momentum tensor, $\mathbf{T} = \rho\,\mathbf{u} \otimes \mathbf{u}$, where ρ is the fluid density and $\mathbf{u}$ is the fluid velocity. In that case, we find that

$$\mathbf{F}_\mathrm{A} = \iint_\mathrm{A} u^1\,u^j\,\mathbf{g}_j\,\mathcal{J}\,\mathrm{d}x^2\,\mathrm{d}x^3 = \iint_\mathrm{A} u^1\,\mathbf{u}\,\mathcal{J}\,\mathrm{d}x^2\,\mathrm{d}x^3. \qquad (4.7.24)$$

Similar expressions can be written for other faces. The surface integral in (4.7.20) is the sum of the integrals over the six faces.

Exercise

4.7.1 Derive an expression for the pure covariant components of the tensor product $\mathbf{W} = \mathbf{T} \cdot \mathbf{S}$.

4.8 Coordinate transformations

Consider a system of contravariant coordinates, (x^1, x^2, x^3), and another system of contravariant coordinates, $(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$. The former can be regarded as functions of the later and *vice versa*, in that a point in space, $\mathbf{x}$, is described by corresponding triplets.

4.8.1 Covariant base vector transformations

Using the chain rule, we find that that the covariant base vectors in the second system are related to those in the first system by

$$\widetilde{\mathbf{g}}_i \equiv \frac{\partial \mathbf{x}}{\partial \widetilde{x}^i} = \frac{\partial \mathbf{x}}{\partial x^j}\,\frac{\partial x^j}{\partial \widetilde{x}^i} = \frac{\partial x^j}{\partial \widetilde{x}^i}\,\mathbf{g}_j = H_{ij}\,\mathbf{g}_j, \qquad (4.8.1)$$

where

$$H_{ij} \equiv \frac{\partial x^j}{\partial \widetilde{x}^i} = \widetilde{\mathbf{g}}_i \cdot \mathbf{g}^j \tag{4.8.2}$$

is a covariant base vector transformation matrix and summation is implied over the repeated index, j.

Conversely, the covariant base vectors in the first system are related to those in the second system by

$$\mathbf{g}_i \equiv \frac{\partial \mathbf{x}}{\partial x^i} = \frac{\partial \mathbf{x}}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial \widetilde{x}^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \widetilde{\mathbf{g}}_j = H^{-1}_{ij} \widetilde{\mathbf{g}}_j, \tag{4.8.3}$$

where

$$H^{-1}_{ij} \equiv \frac{\partial \widetilde{x}^j}{\partial x^i} = \mathbf{g}_i \cdot \widetilde{\mathbf{g}}^j \tag{4.8.4}$$

and a superscript -1 denotes the matrix transpose. We may confirm that

$$H_{ik} \, H^{-1}_{kj} = \frac{\partial x^k}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^k} = \delta_{ij}, \tag{4.8.5}$$

as discussed in Section 2.10.1. The transformation matrix $\mathbf{H}$ is not necessarily orthogonal.

4.8.2 Contravariant base vector transformations

Working in a similar fashion, we find that

$$\widetilde{\mathbf{g}}^i = R_{ij} \, \mathbf{g}^j, \qquad \mathbf{g}^i = R^{-1}_{ij} \, \widetilde{\mathbf{g}}^j, \tag{4.8.6}$$

where

$$R_{ij} = \widetilde{\mathbf{g}}^i \cdot \mathbf{g}_j \qquad R^{-1}_{ij} = \mathbf{g}^i \cdot \widetilde{\mathbf{g}}_j. \tag{4.8.7}$$

We may confirm that

$$R_{ik} R^{-1}_{kj} = \delta_{ij}, \tag{4.8.8}$$

as discussed in Section 2.10.3.

4.8.3 Relation between transformation matrices

Next, we compute

$$\widetilde{\mathbf{g}}_i \cdot \widetilde{\mathbf{g}}^j = (H_{ik}\,\mathbf{g}_k) \cdot (R_{jm}\,\mathbf{g}^m) = H_{ik} R_{jm} \delta_{km} = H_{ik} R_{jk} = \delta_{ij}, \tag{4.8.9}$$

which shows that

$$\mathbf{R} = \mathbf{H}^{-\mathrm{T}}, \qquad \mathbf{H} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{I}, \qquad \mathbf{R} \cdot \mathbf{H}^{\mathrm{T}} = \mathbf{I}. \tag{4.8.10}$$

In summary, we have defined or derived the relations

$$\begin{aligned} \widetilde{\mathbf{g}}_i \cdot \mathbf{g}^j &= H_{ij}, & \mathbf{g}_i \cdot \widetilde{\mathbf{g}}^j &= H_{ij}^{-1}, \\ \widetilde{\mathbf{g}}^i \cdot \mathbf{g}_j &= H_{ij}^{-\mathrm{T}}, & \mathbf{g}^i \cdot \widetilde{\mathbf{g}}_j &= H_{ij}^{\mathrm{T}}. \end{aligned} \tag{4.8.11}$$

4.8.4 Vector components

A vector, $\mathbf{v}$, can be resolved in four ways as

$$\mathbf{v} = v^i\,\mathbf{g}_i = v_i\,\mathbf{g}^i = \widetilde{v}^i\,\widetilde{\mathbf{g}}_i = \widetilde{v}_i\,\widetilde{\mathbf{g}}^i, \tag{4.8.12}$$

where $\widetilde{v}^i$ and $\widetilde{v}_i$ are the contravariant and covariant vector components in the second system denoted by a tilde. Performing projections, we derive relations between the contravariant vector components,

$$\widetilde{v}^i = H_{ji}^{-1} v^j, \qquad v^i = H_{ji} \widetilde{v}^j \tag{4.8.13}$$

and corresponding relations for the covariant vector components,

$$\widetilde{v}_i = H_{ij} v_j, \qquad v_i = H_{ij}^{-1} \widetilde{v}_j. \tag{4.8.14}$$

In terms of coordinate derivatives,

$$\widetilde{v}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\, v^j, \qquad v^i = \frac{\partial x^i}{\partial \widetilde{x}^j}\, \widetilde{v}^j \tag{4.8.15}$$

and

$$\widetilde{v}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\, v_j, \qquad v_i = \frac{\partial \widetilde{x}^j}{\partial x^i}\, \widetilde{v}_j. \tag{4.8.16}$$

Relations (4.8.15) and (4.8.16) are the distinguishing properties of first-order tensors representing physical vectors.

4.8.5 Two-index tensors

In the case of two-index tensors, we derive the relations

$$\widetilde{T}^{ij} = H^{-1}_{pi} H^{-1}_{qj} T^{pq}, \qquad \widetilde{T}_{ij} = H_{ip} H_{jq} T_{pq} \tag{4.8.17}$$

and

$$\widetilde{T}^{i}_{\circ j} = H^{-1}_{pi} H_{jq} T^{p}_{\circ q}, \qquad \widetilde{T}^{\circ j}_{i} = H_{ip} H^{-1}_{qj} T^{\circ q}_{p}. \tag{4.8.18}$$

In terms of coordinate derivatives,

$$\widetilde{T}^{ij} = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} T^{pq}, \qquad \widetilde{T}_{ij} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} T_{pq} \tag{4.8.19}$$

and

$$\widetilde{T}^{i}_{\circ j} = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \widetilde{x}^j} T^{p}_{\circ q}, \qquad \widetilde{T}^{\circ j}_{i} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} T^{\circ q}_{p}, \tag{4.8.20}$$

where summation is implied over the repeated indices, p and q.

4.8.6 Transformation of metric coefficients

Applying the transformation rules (4.8.19) and (4.8.20) for the four sets of metric coefficients, we obtain

$$\widetilde{g}^{ij} = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} g^{pq}, \qquad \widetilde{g}_{ij} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} g_{pq} \tag{4.8.21}$$

and

$$\widetilde{g}^{i}_{\circ j} = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \widetilde{x}^j} g^{p}_{\circ q}, \qquad \widetilde{g}^{\circ j}_{i} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} g^{\circ q}_{p}, \tag{4.8.22}$$

where

$$\widetilde{g}^{i}_{\circ j} = \delta_{ij}, \qquad g^{p}_{\circ q} = \delta_{pq}, \qquad \widetilde{g}^{\circ j}_{i} = \delta_{ij}, \qquad g^{\circ q}_{p} = \delta_{pq}. \tag{4.8.23}$$

4.8.7 Determinants

Based on the transformation rules (4.8.19) and (4.8.20), we derive the following relations among the determinants:

$$\begin{aligned}
\det[\widetilde{T}^{ij}] &= \det[\frac{\partial \widetilde{x}^i}{\partial x^p}] \det[\frac{\partial \widetilde{x}^j}{\partial x^q}] \det[T^{pq}],\\
\det[\widetilde{T}_{ij}] &= \det[\frac{\partial x^p}{\partial \widetilde{x}^i}] \det[\frac{\partial x^q}{\partial \widetilde{x}^j}] \det[T_{pq}],\\
\det[\widetilde{T}^i_{\circ j}] &= \det[\frac{\partial \widetilde{x}^i}{\partial x^p}] \det[\frac{\partial x^q}{\partial \widetilde{x}^j}] \det[T^p_{\circ q}],\\
\det[\widetilde{T}_i^{\circ j}] &= \det[\frac{\partial x^p}{\partial \widetilde{x}^i}] \det[\frac{\partial \widetilde{x}^j}{\partial x^q}] \det[T_p^{\circ q}],
\end{aligned} \tag{4.8.24}$$

where summation is implied over the repeated indices, p and q.

4.8.8 Determinant of a tensor

We can identify the tilde coordinates with the universal Cartesian coordinates by setting

$$\widetilde{x}^1 = x, \quad \widetilde{x}^2 = y, \quad \widetilde{x}^3 = z, \tag{4.8.25}$$

note that

$$\det(\mathbf{T}) = \det[\widetilde{T}^{ij}] = \det[\widetilde{T}_{ij}] = \det[\widetilde{T}^i_{\circ j}] = \det[\widetilde{T}_i^{\circ j}] \tag{4.8.26}$$

and also

$$\det[\frac{\partial \widetilde{x}^i}{\partial x^p}] = \mathcal{J}, \qquad \det[\frac{\partial x^p}{\partial \widetilde{x}^i}] = \frac{1}{\mathcal{J}}, \tag{4.8.27}$$

where $\mathcal{J}^2 = g$, and obtain

$$\det(\mathbf{T}) = g \det[T^{ij}] = \frac{1}{g} \det[T_{ij}] = \det[T_i^{\circ j}] = \det[T^i_{\circ j}]. \tag{4.8.28}$$

These results are consistent with those derived in Section 2.12.6 in a more general context.

4.8.9 High-order tensor components

Working in a similar fashion, we find that tensor components in the two coordinate systems transform according to the general rule

$$\widetilde{T}^{i_1\ldots i_m}_{j_1\ldots j_n} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^{r_1}} \cdots \frac{\partial \widetilde{x}^{i_m}}{\partial x^{r_m}} \, \frac{\partial x^{s_1}}{\partial \widetilde{x}^{j_1}} \cdots \frac{\partial x^{s_n}}{\partial \widetilde{x}^{j_n}} \, T^{r_1\ldots r_m}_{s_1\ldots s_n}, \qquad (4.8.29)$$

where the ordering of the lower and upper indices is immaterial.

Exercise

4.8.1 Derive relation (4.8.29).

4.9 Christoffel symbols

The derivatives of the covariant base vectors, $\mathbf{g}_i$, with respect to the contravariant coordinates, x^j, are vectors themselves. We may define the associated *vectorial* Christoffel symbols,

$$\boldsymbol{\Gamma}_{ij} \equiv \frac{\partial \, \mathbf{g}_i}{\partial x^j} = \frac{\partial^2 \mathbf{x}}{\partial x^i \partial x^j}, \qquad (4.9.1)$$

which are zero only in the case of Cartesian or oblique rectilinear coordinates. Note that $\boldsymbol{\Gamma}_{ij}$ is a vector typeset in bold.

The vectorial Christofell symbols are the second derivatives of the position with respect to contravariant coordinates. Consequently,

$$\boldsymbol{\Gamma}_{ij} = \boldsymbol{\Gamma}_{ji}, \qquad (4.9.2)$$

which shows that the symbols are symmetric with respect to the subscripts i and j.

4.9.1 Christofell symbols of the second kind

The vectorial Christoffel symbols can be expressed in terms of the covariant base vectors as

$$\frac{\partial \, \mathbf{g}_i}{\partial x^j} \equiv \Gamma^k_{ij} \, \mathbf{g}_k, \qquad (4.9.3)$$

where Γ^k_{ij} are the Christoffel symbols of the second kind and summation is implied over the repeated index, k. Thus, by definition,

$$\mathbf{\Gamma}_{ij} \equiv \Gamma^k_{ij}\, \mathbf{g}_k, \tag{4.9.4}$$

where summation is implied over the repeated index, k. The Christoffel symbols of the second kind are identically zero in Cartesian or homogeneous oblique coordinates.

Projecting equation (4.9.3) onto $\mathbf{g}^m$, where m is a free index, using the biorthonormality of the covariant and contravariant base vectors, $\mathbf{g}_k \cdot \mathbf{g}^m = \delta_{km}$, and then switching m to k, we obtain

$$\Gamma^k_{ij} = \frac{\partial\, \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k, \tag{4.9.5}$$

which defines the Christoffel symbols of the second kind in terms of derivatives of the covariant base vectors with respect to contravariant coordinates.

4.9.2 Symmetry

Equation (4.9.4) combined with the symmetry property $\mathbf{\Gamma}_{ij} = \mathbf{\Gamma}_{ji}$, suggests that

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{4.9.6}$$

As a further confirmation, we use the definition of the Christoffel symbol stated in (4.9.5) to write

$$\Gamma^k_{ij} = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k = \frac{\partial^2 \mathbf{x}}{\partial x^i \partial x^j} \cdot \mathbf{g}^k = \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}^k = \Gamma^k_{ji}. \tag{4.9.7}$$

Thus, the Christoffel symbols of the second kind are symmetric with respect to the subscripts i and j.

4.9.3 Not a tensor

It should be noted that the Christoffel symbols of the second kind are *not* components of a tensor. To emphasize this, the following notation is sometimes used,

$$\Gamma^k_{ij} \equiv \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}. \tag{4.9.8}$$

4.9.4 Derivatives of contravariant base vectors

Using the biorthonormality of the covariant and contravariant base vectors, $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_{ik}$, we find that

$$\frac{\partial(\mathbf{g}_i \cdot \mathbf{g}^k)}{\partial x^j} = 0 \tag{4.9.9}$$

for any trio of arbitrary indices, i, k, and j. Expanding the derivative, we obtain

$$-\frac{\partial\, \mathbf{g}^k}{\partial x^j} \cdot \mathbf{g}_i = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k \equiv \Gamma^k_{ij}, \tag{4.9.10}$$

which implies that

$$\frac{\partial\, \mathbf{g}^k}{\partial x^j} = -\Gamma^k_{ij}\, \mathbf{g}^i, \tag{4.9.11}$$

which is a companion of (4.9.3).

4.9.5 Christoffel symbols in terms of the metric tensor

To express the Christoffel symbols of the second kind in terms of the components of the metric tensor, we compute the derivatives

$$\frac{\partial g_{mi}}{\partial x^j} = \frac{\partial(\mathbf{g}_m \cdot \mathbf{g}_i)}{\partial x^j} = \mathbf{g}_m \cdot \frac{\partial \mathbf{g}_i}{\partial x^j} + \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_m}{\partial x^j}, \tag{4.9.12}$$

and thus obtain

$$\frac{\partial g_{mi}}{\partial x^j} = \Gamma^k_{ij}\, \mathbf{g}_m \cdot \mathbf{g}_k + \Gamma^k_{mj}\, \mathbf{g}_i \cdot \mathbf{g}_k, \tag{4.9.13}$$

yielding

$$\frac{\partial g_{mi}}{\partial x^j} = \Gamma^k_{ij}\, g_{mk} + \Gamma^k_{mj}\, g_{ik}, \tag{4.9.14}$$

where summation is implied over the repeated index, k.

Next, we multiply equation (4.9.14) by g^{mp}, where p is a free index, sum over m, and recall that $g_{mk}\, g^{mp} = \delta_{kp}$ to obtain

$$g^{mp}\, \frac{\partial g_{mi}}{\partial x^j} = \Gamma^p_{ij} + \Gamma^k_{mj}\, g_{ik}\, g^{mp}. \tag{4.9.15}$$

Switching the indices i and j, we obtain the companion relationship

$$g^{mp}\,\frac{\partial g_{mj}}{\partial x^i} = \Gamma^p_{ij} + \Gamma^k_{mi}\,g_{jk}\,g^{mp}. \tag{4.9.16}$$

Adding equations (4.9.15) and (4.9.16), we obtain

$$g^{mp}\,\frac{\partial g_{mi}}{\partial x^j} + g^{mp}\,\frac{\partial g_{mj}}{\partial x^i} = 2\,\Gamma^p_{ij} + g^{mp}\big(\Gamma^k_{mj}\,g_{ik} + \Gamma^k_{mi}\,g_{jk}\big). \tag{4.9.17}$$

Combining equations (4.9.14) and (4.9.17), we find that the Christoffel symbols of the second kind can be obtained from the contravariant and covariant components of the metric tensor using the expression

$$\Gamma^p_{ij} = \tfrac{1}{2}\,g^{pm}\Big(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\Big), \tag{4.9.18}$$

where summation is implied over the repeated index, m. Equation (4.9.18) is known as the Levi–Civita connection.

4.9.6 Coordinate transformations

Consider a system of curvilinear coordinates, (x^1, x^2, x^3), and another system of curvilinear coordinates, $(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$. Departing from the second transformation in (4.8.21), we write

$$\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{x}^k} = \frac{\partial}{\partial \widetilde{x}^k}\big(\,\frac{\partial x^p}{\partial \widetilde{x}^i}\,\frac{\partial x^q}{\partial \widetilde{x}^j}\,g_{pq}\,\big) \tag{4.9.19}$$

and then

$$\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{x}^k} = \big(\,\frac{\partial^2 x^p}{\partial \widetilde{x}^k \partial \widetilde{x}^i}\,\frac{\partial x^q}{\partial \widetilde{x}^j} + \frac{\partial^2 x^p}{\partial \widetilde{x}^k \partial \widetilde{x}^j}\,\frac{\partial x^q}{\partial \widetilde{x}^i}\,\big)\,g_{pq} + S, \tag{4.9.20}$$

where

$$S \equiv \frac{\partial x^p}{\partial \widetilde{x}^i}\,\frac{\partial x^q}{\partial \widetilde{x}^j}\,\frac{\partial g_{pq}}{\partial \widetilde{x}^k} = \frac{\partial x^p}{\partial \widetilde{x}^i}\,\frac{\partial x^q}{\partial \widetilde{x}^j}\,\frac{\partial x^m}{\partial \widetilde{x}^k}\,\frac{\partial g_{pq}}{\partial x^m}. \tag{4.9.21}$$

The second expression for S arises by using the chain rule. Substituting this expansion into the counterpart of (4.9.18) for the tilded variables and simplifying, we obtain

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^p}\,\frac{\partial x^r}{\partial \widetilde{x}^i}\,\frac{\partial x^s}{\partial \widetilde{x}^j}\,\Gamma^p_{rs} + \frac{\partial^2 x^p}{\partial \widetilde{x}^i \partial \widetilde{x}^j}\,\frac{\partial \widetilde{x}^k}{\partial x^p}, \tag{4.9.22}$$

where summation is implied over the repeated indices, p, r, s. This formula will be derived in Section 5.5.7 using a different method.

Multiplying expression (4.9.22) by $\partial x^m / \partial \widetilde{x}^k$, recalling that

$$\frac{\partial \widetilde{x}^k}{\partial x^p} \frac{\partial x^m}{\partial \widetilde{x}^k} = \delta_{pm}, \tag{4.9.23}$$

as required for two inverse functions, and solving for the second term on the right-hand side, we obtain the Christoffel formula

$$\frac{\partial^2 x^m}{\partial \widetilde{x}^i \partial \widetilde{x}^j} = \frac{\partial x^m}{\partial \widetilde{x}^k} \widetilde{\Gamma}^k_{ij} - \frac{\partial x^r}{\partial \widetilde{x}^i} \frac{\partial x^s}{\partial \widetilde{x}^j} \Gamma^m_{rs}, \tag{4.9.24}$$

where summation is implied over the repeated indices, k, r, s.

4.9.7 Alternative expression

Another expression for the Christoffel symbols can be derived. Recalling that

$$\frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^k}{\partial x^p} = \delta_{ik}, \tag{4.9.25}$$

differentiating with respect to $\widetilde{x}^j$, expanding the derivative of the product, setting the derivative of the right-hand side to zero, and applying the chain rule, we obtain

$$\frac{\partial^2 x^p}{\partial \widetilde{x}^i \widetilde{x}^j} \frac{\partial \widetilde{x}^k}{\partial x^p} = -\frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial}{\partial \widetilde{x}^j} \Big(\frac{\partial \widetilde{x}^k}{\partial x^p} \Big) = -\frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \frac{\partial}{\partial x^q} \Big(\frac{\partial \widetilde{x}^k}{\partial x^p} \Big) \tag{4.9.26}$$

and then

$$\frac{\partial^2 x^p}{\partial \widetilde{x}^i \widetilde{x}^j} \frac{\partial \widetilde{x}^k}{\partial x^p} = -\frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^r}{\partial \widetilde{x}^j} \frac{\partial^2 \widetilde{x}^k}{\partial x^p \partial x^r}. \tag{4.9.27}$$

Substituting this property into the transformation rule (4.9.22), we obtain

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^p} \frac{\partial x^r}{\partial \widetilde{x}^i} \frac{\partial x^s}{\partial \widetilde{x}^j} \Gamma^p_{rs} - \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^r}{\partial \widetilde{x}^j} \frac{\partial^2 \widetilde{x}^k}{\partial x^p \partial x^r}, \tag{4.9.28}$$

where summation is implied over the repeated indices, p, r, s. This formula will be derived in Section 5.5.8 using a different method.

4.9.8 From Cartesian to curvilinear

Now identifying the untilded coordinates with Cartesian coordinates indicated by Greek subscripts, and setting the corresponding Christoffel symbols to zero, we find that

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial^2 x_\alpha}{\partial \widetilde{x}^i \partial \widetilde{x}^j} \, \frac{\partial \widetilde{x}^k}{\partial x_\alpha} = -\frac{\partial x_\alpha}{\partial \widetilde{x}^i} \frac{\partial x_\beta}{\partial \widetilde{x}^j} \, \frac{\partial^2 \widetilde{x}^k}{\partial x_\alpha \partial x_\beta}, \tag{4.9.29}$$

where summation is implied over α and β in the range $1, 2, 3$.

4.9.9 Christoffel symbols of the first kind

We recall the expression for the Christoffel symbols of the second kind given in (4.9.18). Multiplying this equation by g_{qp}, where q is a free index, and summing over p, we obtain the Christoffel symbol of the first kind,

$$\Gamma_{q;ij} \equiv g_{qp} \, \Gamma^p_{ij}. \tag{4.9.30}$$

This equation is deceptive in that $\Gamma_{q;ij}$ might falsely appear as the pure covariant counterpart of Γ^p_{ij}.

Explicitly, the Christoffel symbol of the first kind is given by

$$\Gamma_{q;ij} = \tfrac{1}{2} \, g^{pm} g_{qp} \left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right). \tag{4.9.31}$$

Recalling that $g^{pm} g_{qp} = \delta_{mq}$, we obtain

$$\Gamma_{q;ij} \equiv g_{qk} \, \Gamma^p_{ij} = \tfrac{1}{2} \left(\frac{\partial g_{qi}}{\partial x^j} + \frac{\partial g_{qj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^q} \right). \tag{4.9.32}$$

The preceding expressions show that the Christoffel symbol of the first kind is symmetric with respect to the indices i and j,

$$\Gamma_{q;ij} = \Gamma_{q;ji}. \tag{4.9.33}$$

Note the semi-colon (;) notation.

Using expression (4.9.32), we find that equation (4.9.18) becomes

$$\Gamma^p_{ij} = g^{pm} \, \Gamma_{m;ij}, \tag{4.9.34}$$

which provides us with a method of figuratively raising the subscript of the Christoffel symbols of the second kind.

Combining equation (4.9.30) with Ricci's lemma stated in (5.6.19), repeated below for convenience,

$$\frac{\partial g_{ik}}{\partial x^j} = g_{kp}\,\Gamma^p_{ij} + g_{ip}\,\Gamma^p_{jk}, \tag{4.9.35}$$

we obtain

$$\frac{\partial g_{ik}}{\partial x^j} = \Gamma_{k;ij} + \Gamma_{i;kj}. \tag{4.9.36}$$

In practice, the Christoffel symbols of the first kind provide us with a venue for obtaining the Christoffel symbols of the second kind in terms of the components of the metric tensor.

Exercises

4.9.1 Compute the Christofell symbols of the first kind on cylindrical polar coordinates.

4.9.2 Derive (4.9.22).

4.10 Cylindrical polar coordinates

An arbitrary point in space can be identified by its cylindrical polar coordinates, (φ, x, σ), where φ is the azimuthal angle, x is the axial position, and σ is the distance from the x axis as illustrated in Figure 4.10.1. The triplet, (φ, x, σ), comprise orthogonal curvilinear coordinates

$$x^1 = \varphi, \qquad x^2 = x, \qquad x^3 = \sigma, \tag{4.10.1}$$

where $0 \leq \varphi < 2\pi$ by convention, x is arbitrary, and $\sigma \geq 0$. Cyclic permutation of these variables may also be employed.

The Cartesian coordinates of the position vector are

$$y = \sigma \cos\varphi, \qquad z = \sigma \sin\varphi. \tag{4.10.2}$$

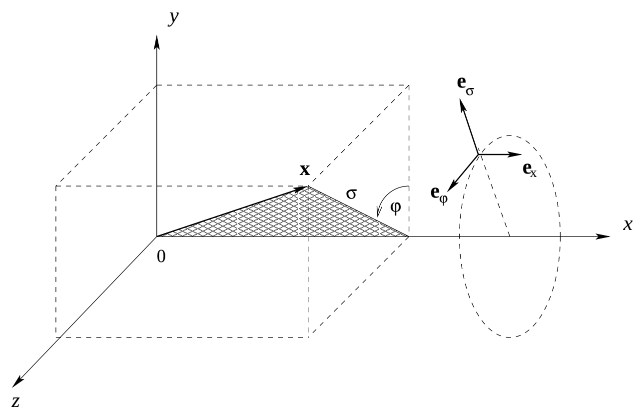

FIGURE 4.10.1 Illustration of cylindrical polar coordinates, (φ, x, σ), defined with respect to Cartesian coordinates, (x, y, z), where σ is the distance from the x axis.

These relations can be inverted readily to yield

$$\sigma = \sqrt{y^2 + z^2}, \qquad \varphi = \arccos \frac{y}{\sigma}, \tag{4.10.3}$$

for any x.

The base unit vectors are

$$\mathbf{e}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}, \quad \mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_\sigma = \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix}, \tag{4.10.4}$$

the covariant base vectors are

$$\mathbf{g}_\varphi = \sigma\, \mathbf{e}_\varphi, \qquad \mathbf{g}_x = \mathbf{e}_x, \qquad \mathbf{g}_\sigma = \mathbf{e}_\sigma, \tag{4.10.5}$$

and the contravariant base vectors are

$$\mathbf{g}^\varphi = \frac{1}{\sigma}\, \mathbf{e}_\varphi, \qquad \mathbf{g}^x = \mathbf{e}_x, \qquad \mathbf{g}^\sigma = \mathbf{e}_\sigma. \tag{4.10.6}$$

Because the cylindrical polar coordinates are orthogonal, the contravariant base vectors are parallel to the corresponding covariant vectors. Note that $\mathbf{g}_\varphi \cdot \mathbf{g}^\varphi = 1$, as required.

4.10.1 Metric coefficients

All contravariant and covariant metric coefficients are zero, $g_{ij} = 0$ and $g^{ij} = 0$, except for the diagonal coefficients

$$g_{\varphi\varphi} = \sigma^2, \qquad g_{xx} = 1, \qquad g_{\sigma\sigma} = 1, \tag{4.10.7}$$

and

$$g_{\varphi\varphi} = \frac{1}{\sigma^2}, \qquad g_{xx} = 1, \qquad g_{\sigma\sigma} = 1. \tag{4.10.8}$$

The non-vanishing of the diagonal components is typical of orthogonal coordinates. Note that $g^{ii} = 1/g_{ii}$, where summation is *not* implied over the repeated index, i.

4.10.2 Christoffel symbols

Using (4.9.5), we compute the Christoffel symbols of the second kind

$$\Gamma^{\varphi}_{\sigma\varphi} = \Gamma^{\varphi}_{\varphi\sigma} = \frac{\partial \mathbf{g}_\sigma}{\partial \varphi} \cdot \mathbf{g}^{\varphi} = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix} \cdot \frac{1}{\sigma} \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix} = \frac{1}{\sigma} \tag{4.10.9}$$

and

$$\Gamma^{\sigma}_{\varphi\varphi} = \frac{\partial \mathbf{g}_\varphi}{\partial \varphi} \cdot \mathbf{g}^{\sigma} = \sigma \begin{bmatrix} 0 \\ -\cos\varphi \\ -\sin\varphi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix} = -\sigma. \tag{4.10.10}$$

Alternatively, expressions (4.10.10) and (4.10.10) can be derived from the transformation rule stated in (4.9.29). All other Christoffel symbols of the second kind turn out to be zero.

Exercises

4.10.1 Confirm that $\Gamma^{\varphi}_{\sigma\varphi} = \Gamma^{\varphi}_{\varphi\sigma}$.

4.10.2 Derive expressions (4.10.10) and (4.10.10) from the transformation rule stated in (4.9.29).

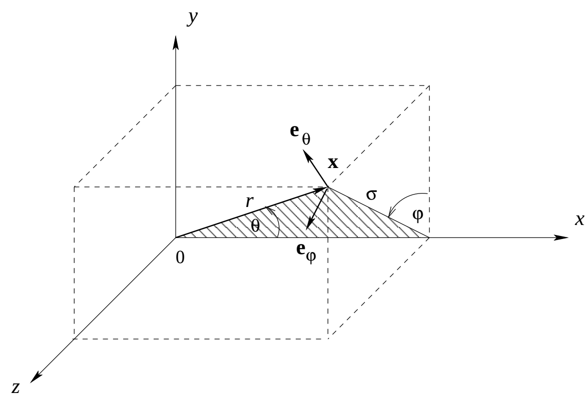

FIGURE 4.11.1 Illustration of spherical polar coordinates, (r, θ, φ), defined with respect to the Cartesian coordinates, (x, y, z), and cylindrical polar coordinates, (x, σ, φ), where r is the distance from the origin, θ is the meridional angle, φ is the azimuthal angle, and σ is the distance from the x axis.

4.11 Spherical polar coordinates

An arbitrary point in space can be identified by its spherical polar coordinates, (θ, φ, r), where r is the distance from the origin, θ is the meridional angle, and φ is the azimuthal angle, as illustrated in Figure 4.11.1. The triplet, (θ, φ, r), comprise orthogonal curvilinear coordinates,

$$x^1 = \theta, \qquad x^2 = \varphi, \qquad x^3 = r, \tag{4.11.1}$$

where $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$, and $r \ge 0$.

The Cartesian coordinates of the position vector are

$$x = r\cos\theta, \qquad y = r\sin\theta\cos\varphi, \qquad z = r\sin\theta\sin\varphi. \tag{4.11.2}$$

Inverting these relations, we obtain the coordinates r, θ, and ϕ in terms of x, y, and z,

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arccos\frac{x}{r}, \qquad \varphi = \arccos\frac{y}{\sigma}. \tag{4.11.3}$$

The base unit vectors are

$$\mathbf{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \end{bmatrix}, \qquad \mathbf{e}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix},$$
$$\mathbf{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix}. \tag{4.11.4}$$

The corresponding covariant base vectors are

$$\mathbf{g}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\,\mathbf{e}_\theta, \qquad \mathbf{g}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = r\,\sin\theta\,\mathbf{e}_\varphi,$$
$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r} = \mathbf{e}_r. \tag{4.11.5}$$

The associated contravariant vectors are given by

$$\mathbf{g}^\theta = \frac{1}{r}\,\mathbf{e}_\theta, \qquad \mathbf{g}^\varphi = \frac{1}{r\sin\theta}\,\mathbf{e}_\varphi. \qquad \mathbf{g}^r = \mathbf{e}_r. \tag{4.11.6}$$

Because the spherical polar coordinates are orthogonal, the contravariant base vectors are parallel to the corresponding covariant vectors.

4.11.1 Metric coefficients

All contravariant and covariant metric coefficients are zero, $g_{ij} = 0$ and $g^{ij} = 0$, except for the diagonal components

$$g_{\theta\theta} = r^2, \qquad g_{\varphi\varphi} = r^2 \sin^2\theta, \qquad g_{rr} = 1, \tag{4.11.7}$$

and

$$g^{\theta\theta} = \frac{1}{r^2}, \qquad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}, \qquad g^{rr} = 1. \tag{4.11.8}$$

The non-vanishing of the diagonal components is typical of orthogonal coordinates. Note that $g^{ii} = 1/g_{ii}$, where summation is *not* implied over the repeated index, i. The volume metric coefficient is given by $\mathcal{J} = r^2 \sin\theta$.

4.11.2 Christoffel symbols

Using (4.9.5), we find that

$$\Gamma^{\varphi}_{r\varphi} = \frac{\partial\,\mathbf{g}_r}{\partial\varphi}\cdot\mathbf{g}^{\varphi} = \begin{bmatrix} 0 \\ -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \end{bmatrix}\cdot\frac{1}{r\sin\theta}\begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix} = \frac{1}{r}. \tag{4.11.9}$$

Working in a similar fashion , we find that all other Christoffel symbols of the second kind are zero, except for the following:

$$\begin{aligned} &\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \\ &\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta, \qquad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \\ &\Gamma^{r}_{\theta\theta} = -r, \qquad \Gamma^{r}_{\varphi\varphi} = -r\,\sin^2\theta. \end{aligned} \tag{4.11.10}$$

Note that the symmetry property with respect to the indices of the Christoffel symbols is satisfied.

Exercise

4.11.1 Derive the non-zero Christoffel symbols shown in (4.11.10).

4.12 Helical coordinates

Helical coordinates are employed when the structure of a scalar or vector field of interest is invariant along a helical path. Examples include fluid flow in a tube with helical corrugations, flow through a tube with a helical centerline, and flow induced by a helical line vortex. In these applications, it is convenient to identity a point in space by the non-orthogonal helical coordinates, $(\hat{\varphi}, \hat{x}, \hat{\sigma})$, defined in Figure 4.12.1.

4.12.1 Relation to cylindrical polar coordinates

The helical coordinates are related to the cylindrical polar coordinates, (x, σ, φ), by

$$\varphi = \hat{\varphi} + \alpha\,\hat{x}, \qquad x = \hat{x}, \qquad \sigma = \hat{\sigma}, \tag{4.12.1}$$

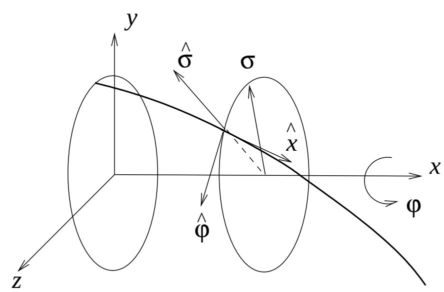

FIGURE 4.12.1 Illustration of helical coordinates, $(\hat{\varphi}, \hat{x}, \hat{\sigma})$, in relation to companion Cartesian and cylindrical polar coordinates, (x, σ, φ).

and to the associated Cartesian coordinates by

$$x = \hat{x}, \qquad y = \hat{\sigma}\,\cos(\hat{\varphi} + \alpha\,\hat{x}), \qquad z = \hat{\sigma}\,\sin(\hat{\varphi} + \alpha\,\hat{x}), \tag{4.12.2}$$

where $\alpha \equiv 2\pi/L$ is the helical wave number and L is the helical pitch. In the limit of infinite pitch, $\alpha \to 0$, the helical coordinates reduce to cylindrical polar coordinates. The variable $\hat{\varphi}$ represents the azimuthal angle in the yz plane at a certain axial position, x.

In problems with helical symmetry, the partial derivative of a variable of interest, f, with respect to $\hat{x}$ is zero, so that $f(\hat{\sigma}, \hat{\varphi})$. To conform with standard notation, we set

$$x^1 = \hat{\varphi}, \qquad x^2 = \widehat{x}, \qquad x^3 = \hat{\sigma}. \tag{4.12.3}$$

The first base vector is given by

$$\mathbf{g}_1 \equiv \frac{\partial \mathbf{x}}{\partial x^1} = \frac{\partial \mathbf{x}}{\partial \hat{\varphi}} = \hat{\sigma} \begin{bmatrix} 0 \\ -\sin(\hat{\varphi} + \alpha\,\hat{x}) \\ \cos(\hat{\varphi} + \alpha\,\hat{x}) \end{bmatrix}, \tag{4.12.4}$$

the second base vector is given by

$$\mathbf{g}_2 \equiv \frac{\partial \mathbf{x}}{\partial x^2} = \frac{\partial \mathbf{x}}{\partial \hat{x}} = \begin{bmatrix} 1 \\ -\alpha\hat{\sigma}\sin(\hat{\varphi} + \alpha\,\hat{x}) \\ \alpha\hat{\sigma}\cos(\hat{\varphi} + \alpha\,\hat{x}) \end{bmatrix}, \tag{4.12.5}$$

and the third base vector is given by

$$\mathbf{g}_3 \equiv \frac{\partial \mathbf{x}}{\partial x^3} = \frac{\partial \mathbf{x}}{\partial \hat{\sigma}} = \begin{bmatrix} 0 \\ \cos(\hat{\varphi} + \alpha\,\hat{x}) \\ \sin(\hat{\varphi} + \alpha\,\hat{x}) \end{bmatrix}. \tag{4.12.6}$$

4.12.2 Metric coefficients

The matrix of covariant metric coefficients is given by

$$\mathbf{g} \equiv \big[\, g_{ij} \,\big] = \begin{bmatrix} \hat{\sigma}^2 & \alpha\,\hat{\sigma}^2 & 0 \\ \alpha\,\hat{\sigma}^2 & 1 + \alpha^2\,\hat{\sigma}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.12.7}$$

The matrix of contravariant metric coefficients is the inverse of the matrix of covariant metric coefficients,

$$\boldsymbol{\beta} \equiv \big[\, g^{ij} \,\big] = \begin{bmatrix} \dfrac{1}{\hat{\sigma}^2} + \alpha^2 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.12.8}$$

The presence of nondiagonal elements indicates that the helical coordinates are orthogonal only in the limit of infinite pitch, $\alpha \to 0$.

4.12.3 Christoffel symbols

The only non-zero Christoffel symbols are the following:

$$\Gamma^3_{32} = \frac{1}{\hat{\sigma}}, \qquad \Gamma^3_{12} = \frac{\alpha}{\hat{\sigma}}, \qquad \Gamma^2_{33} = -\hat{\sigma}, \qquad \Gamma^2_{13} = -\alpha\,\hat{\sigma},$$
$$\Gamma^3_{23} = \frac{1}{\hat{\sigma}}, \qquad \Gamma^2_{31} = -\alpha\,\hat{\sigma}, \qquad \Gamma^2_{11} = -\alpha^2\,\hat{\sigma}, \qquad \Gamma^3_{21} = \frac{\alpha}{\hat{\sigma}}. \tag{4.12.9}$$

4.12.4 Alternative helical coordinates

Alternative helical coordinates $(\tilde{\varphi}, \tilde{x}, \tilde{\sigma})$ can be defined such that the cylindrical polar coordinates are $\varphi = \tilde{\varphi}$, $x = \tilde{x} + \frac{1}{\alpha}\,\tilde{\varphi}$, and $\sigma = \tilde{\sigma}$. The associated Cartesian coordinates are

$$x = \tilde{x} + \frac{1}{\alpha}\,\tilde{\varphi}, \qquad y = \tilde{\sigma}\,\cos\tilde{\varphi}, \qquad z = \tilde{\sigma}\,\sin\tilde{\varphi}. \tag{4.12.10}$$

The base vectors and metric coefficients can be computed by straightforward differentiation.

Exercises

4.12.1 Compute the covariant base vectors corresponding to (4.12.10).

4.12.2 Confirm that the matrix given in (4.12.7) is the inverse of that given in (4.12.8).

4.13 Covariant derivatives of vector components

The derivative of a vector field, $\mathbf{v}$, with respect to a contravariant coordinate, x^j, is another vector field that can be expressed in terms of properly defined covariant derivatives of the contravariant or covariant vector components, v^i or v_i, in two combinations. The covariant derivatives are defined in terms of the Christoffel symbols introduced in Section 4.8.

4.13.1 Covariant derivative of contravariant vector components

We may expand $\mathbf{v} = v^i \mathbf{g}_i$ and take the derivative with respect to the jth contravariant coordinate,

$$\frac{\partial \mathbf{v}}{\partial x^j} = \frac{\partial (v^i \mathbf{g}_i)}{\partial x^j} = \frac{\partial v^i}{\partial x^j}\,\mathbf{g}_i + v^i\,\frac{\partial \mathbf{g}_i}{\partial x^j}, \tag{4.13.1}$$

where summation is implied over the repeated index, i. Expressing the derivative $\partial \mathbf{g}_i / \partial x^j$ in terms of the Christoffel symbols of the second kind by way of the definition (4.9.3), we obtain

$$\frac{\partial \mathbf{v}}{\partial x^j} = \frac{\partial v^i}{\partial x^j}\,\mathbf{g}_i + v^i\,\Gamma^k_{ij}\,\mathbf{g}_k. \tag{4.13.2}$$

Renaming the indices in the second term on the right-hand side and rearranging, we obtain

$$\frac{\partial \mathbf{v}}{\partial x^j} = \Big(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k\Big)\,\mathbf{g}_i. \tag{4.13.3}$$

The expression inside the parentheses on the right-hand side is the covariant derivative of a contravariant vector component, denoted by a comma,

$$\frac{\partial \mathbf{v}}{\partial x^j} = v^i_{,j}\, \mathbf{g}_i, \tag{4.13.4}$$

where summation is implied over the repeated index, i. By definition,

$$v^i_{,j} \equiv \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk}\, v^k. \tag{4.13.5}$$

Other non-comma notations for the covariant derivative, including a vertical dash, are employed in the literature. If the Christofell symbols are all zero, the covariant derivative reduces to the familiar partial derivative.

4.13.2 *Covariant derivative of covariant vector components*

The corresponding representation of the derivative under consideration in terms of contravariant base vectors is

$$\frac{\partial \mathbf{v}}{\partial x^j} = v_{i,j}\, \mathbf{g}^i, \tag{4.13.6}$$

where $v_{i,j}$ is the covariant derivative of the covariant vector components. Using the rule for lowering an index, we find that

$$v_{i,j} = g_{im} v^m_{,j} = g_{im} \left(\frac{\partial v^m}{\partial x^j} + \Gamma^m_{jk}\, v^k \right). \tag{4.13.7}$$

This expression can be simplified by manipulating the first product on the right-hand side and mutually renaming the indices k and m, to obtain

$$v_{i,j} = \frac{\partial (v^m g_{im})}{\partial x^j} - \left(\frac{\partial g_{im}}{\partial x^j} - \Gamma^k_{jm}\, g_{ik} \right) v^m. \tag{4.13.8}$$

Using expression (4.9.14), repeated below for convenience,

$$\frac{\partial g_{mi}}{\partial x^j} = \Gamma^k_{ij}\, g_{mk} + \Gamma^k_{mj}\, g_{ik}, \tag{4.13.9}$$

and simplifying, we obtain

$$v_{i,j} = \frac{\partial v_i}{\partial x^j} - \Gamma^k_{ij}\, g_{mk}\, v^m. \tag{4.13.10}$$

Simplifying further, we derive an expression for the covariant derivative of the covariant vector components,

$$v_{i,j} \equiv \frac{\partial v_i}{\partial x^j} - \Gamma^k_{ji}\, v_k. \tag{4.13.11}$$

The expression differs from that shown in (4.13.5).

Alternatively, expression (4.13.11) can be derived by expanding

$$\frac{\partial \mathbf{v}}{\partial x^j} = \frac{\partial (v_i \mathbf{g}^i)}{\partial x^j} = \frac{\partial v_i}{\partial x^j}\,\mathbf{g}^i + \frac{\partial \mathbf{g}^i}{\partial x^j}\, v_i, \tag{4.13.12}$$

and using (4.9.11) to write

$$\frac{\partial\, \mathbf{g}^i}{\partial x^j} = -\Gamma^i_{kj}\, \mathbf{g}^k. \tag{4.13.13}$$

Renaming the indices we obtain the expression for the covariant derivative shown in (4.13.11).

Exercise

4.13.1 Discuss the notion of a contravariant derivative, that is, a derivative with respect to x^j, and explain its absence from the literature.

4.14 Covariant derivatives of tensor components

The derivative of a tensor field, $\mathbf{T}$, with respect to a contravariant coordinate, x^j, is another tensor field that can be expressed in terms of a properly defined covariant derivative of the contravariant, covariant, or mixed tensor components in four combinations.

4.14.1 Covariant derivative of contravariant tensor components

An arbitrary tensor, $\mathbf{T}$, admits the four-fold expansion

$$\mathbf{T} = T^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j = T^{i}_{\circ j}\,\mathbf{g}_i \otimes \mathbf{g}^j = T_i^{\circ j}\,\mathbf{g}^i \otimes \mathbf{g}_j = T_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j. \tag{4.14.1}$$

Taking the derivative of the first expansion with respect to the kth contravariant coordinate, we find that

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\,\frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\,\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}. \tag{4.14.2}$$

The last two derivatives, $\partial \mathbf{g}_i/\partial x^k$ and $\partial \mathbf{g}_j/\partial x^k$, can be expressed in terms of the Christoffel symbols of the second kind using the definition (4.9.3),

$$\frac{\partial\,\mathbf{g}_i}{\partial x^k} \equiv \Gamma^m_{ik}\,\mathbf{g}_m, \qquad \frac{\partial\,\mathbf{g}_j}{\partial x^k} \equiv \Gamma^m_{jk}\,\mathbf{g}_m. \tag{4.14.3}$$

Substituting these expressions into (5.6.5), we obtain

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\,\Gamma^m_{ik}\,\mathbf{g}_m \otimes \mathbf{g}_j + T^{ij}\,\Gamma^m_{jk}\,\mathbf{g}_i \otimes \mathbf{g}_m. \tag{4.14.4}$$

Next, we mutually rename the indices i and m in the penultimate term and the indices j and m in the last term on the right-hand side, and obtain

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}_i \otimes \mathbf{g}_j + T^{mj}\,\Gamma^i_{mk}\,\mathbf{g}_i \otimes \mathbf{g}_j + T^{im}\,\Gamma^j_{mk}\,\mathbf{g}_i \otimes \mathbf{g}_j. \tag{4.14.5}$$

We have found that

$$\frac{\partial \mathbf{T}}{\partial x^k} = T^{ij}_{,k}\,\mathbf{g}_i \otimes \mathbf{g}_j, \tag{4.14.6}$$

where

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\,T^{mj} + \Gamma^j_{mk}\,T^{im} \tag{4.14.7}$$

is the covariant derivative of the contravariant components of $\mathbf{T}$; summation is implied over the repeated index, m. If the Christofell symbols are all zero, the covariant derivative reduces to the familiar partial derivative.

4.14.2 Covariant derivatives of covariant components

Working as previously in this section with the last expression in (4.14.1), we find that

$$\frac{\partial \mathbf{T}}{\partial x^k} = T_{ij,k}\, \mathbf{g}^i \otimes \mathbf{g}^j, \tag{4.14.8}$$

where

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}\, T_{mj} - \Gamma^m_{kj}\, T_{im} \tag{4.14.9}$$

is the covariant derivative of the covariant components.

4.14.3 Covariant derivatives of con-cov components

Working as previously in this section with the second expression in (4.14.1), we find that

$$\frac{\partial \mathbf{T}}{\partial x^k} = T^i_{\circ j,k}\, \mathbf{g}_i \otimes \mathbf{g}^j, \tag{4.14.10}$$

where

$$T^i_{\circ j,k} = \frac{\partial T^i_{\circ j}}{\partial x^k} + \Gamma^i_{mk}\, T^m_{\circ j} - \Gamma^m_{jk}\, T^i_{\circ m} \tag{4.14.11}$$

is a covariant derivative of mixed components.

4.14.4 Covariant derivatives of cov-con components

Working as previously in this section with the third expression in (5.6.4), we find that

$$\frac{\partial \mathbf{T}}{\partial x^k} = T^{\circ j}_{i,k}\, \mathbf{g}^i \otimes \mathbf{g}_j, \tag{4.14.12}$$

where

$$T^{\circ j}_{i,k} = \frac{\partial T^{\circ j}_{i}}{\partial x^k} - \Gamma^m_{ik}\, T^{\circ j}_{m} + \Gamma^j_{km}\, T^{\circ m}_{i} \tag{4.14.13}$$

is yet another covariant derivative of mixed components.

4.14.5 Mnemonic rule

Note that the sign of the terms involving the Christoffel symbols on the right-hand sides of (4.14.7), (4.14.9), (4.14.11), and (4.14.13) is positive when m appears as a superscript and negative when m appears as a subscript on T.

Exercise

4.14.1 Derive expression (4.14.13).

4.15 Alternating tensor

Previously in this chapter, we discussed two-index tensors defined in terms of dyadic products. Three- and higher-index tensors are defined in a similar fashion.

The alternating tensor is a three-index tensor described by an eight-fold expansion,

$$\boldsymbol{\xi} = \xi^{ijk}\,\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k = \xi_i^{\circ jk}\,\mathbf{g}^i \otimes \mathbf{g}_j \otimes \mathbf{g}_k = \cdots$$
$$= \xi^i_{\circ jk}\,\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k = \xi_{ijk}\,\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k. \qquad (4.15.1)$$

The components of the alternating tensor are given by

$$\xi^{ijk} = [\,\mathbf{g}^i, \mathbf{g}^j, \mathbf{g}^k\,], \quad \xi_i^{\circ jk} = [\,\mathbf{g}_i, \mathbf{g}^j, \mathbf{g}^k\,], \quad \ldots,$$
$$\xi^i_{\circ jk} = [\,\mathbf{g}^i, \mathbf{g}_j, \mathbf{g}_k\,], \quad \xi_{ijk} = [\,\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\,], \qquad (4.15.2)$$

where

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] \equiv (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \qquad (4.15.3)$$

is the tripled mixed product representing the volume of the parallelepiped whose edges are three arbitrary vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Cyclic permutation of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, preserves the triple mixed product. Non-cyclic permutation preserves the magnitude but changes the sign.

4.15.1 Contravariant and covariant components

The pure contravariant and pure covariant components of the alternating tensor are given by

$$\xi^{ijk} = \frac{1}{\mathcal{J}}\,\epsilon_{ijk}, \qquad \xi_{ijk} = \mathcal{J}\,\epsilon_{ijk}, \tag{4.15.4}$$

where $\mathcal{J}$ is the Jacobian metric and ϵ_{ijk} is the Levi–Civita symbol. Consequently,

$$\boldsymbol{\xi} = \frac{1}{\mathcal{J}}\,\epsilon_{ijk}\,\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k = \mathcal{J}\,\epsilon_{ijk}\,\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k. \tag{4.15.5}$$

4.15.2 Cross product

The cross product of two vectors, $\mathbf{v}$ and $\mathbf{u}$, is given by

$$\mathbf{w} \equiv \mathbf{v} \times \mathbf{u} = \boldsymbol{\xi} : (\mathbf{v} \otimes \mathbf{u}), \tag{4.15.6}$$

as discussed in Section 2.13.

Exercise

4.15.1 Prove (4.15.6).

Chapter 5

Vector and tensor calculus

Expressions for partial derivatives, directional derivatives, the divergence, the curl, the gradient, and other differential operators acting on scalar, vector, and tensor fields can be derived in terms of the Christoffel symbols. These expressions may then be substituted into the equations of mathematical physics to provide us with governing equations in non-Cartesian, rectilinear or curvilinear coordinates.

While the general procedures are straightforward, subtleties arise in the case of moving or convected coordinates employed when the governing equations involve an intrinsic velocity field. The advantage of using convected coordinates will be demonstrated and expressions for the Green's function of the convection–diffusion equation will be derived.

5.1 Gradient of a scalar function

Consider a scalar function of position, $f(\mathbf{x})$. The gradient of this function, denoted by $\boldsymbol{\nabla} f$, is a vector pointing in the direction of maximum rate of change of f with respect to directional arc length. For example, if $f(\mathbf{x})$ is a temperature field, then $\boldsymbol{\nabla} f$ is aligned in the direction when the temperature increases the most at a point.

5.1.1 Directional derivative

The projection of the gradient, $\boldsymbol{\nabla} f$ onto a unit vector, $\mathbf{e}$, is the rate of change of f with respect to arc length, ℓ, measured in the direction of the unit vector,

$$\mathbf{e} \cdot \boldsymbol{\nabla} f = e_x \frac{\partial f}{\partial x} + e_y \frac{\partial f}{\partial y} + e_z \frac{\partial f}{\partial z} \tag{5.1.1}$$

or

$$\mathbf{e} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial \ell} = \cos\theta \, |\boldsymbol{\nabla} f|, \tag{5.1.2}$$

where $e_x^2 + e_y^2 + e_z^2 = 1$, and θ is the angle subtended between $\mathbf{e}$ and $\boldsymbol{\nabla} f$. We conclude that $\mathbf{e} \cdot \boldsymbol{\nabla} f$ is maximum when $\theta = 0$, minimum when $\theta = \pi$, and zero when $\theta = \frac{1}{2}\pi$.

If $\mathbf{a}$ is an arbitrary vector, then

$$\mathbf{a} \cdot \boldsymbol{\nabla} f = a_x \frac{\partial f}{\partial x} + a_y \frac{\partial f}{\partial y} + a_z \frac{\partial f}{\partial z} = \cos\theta \, |\mathbf{a}| \, |\boldsymbol{\nabla} f|, \tag{5.1.3}$$

where $|\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2 = 1$ and θ is the angle subtended between $\mathbf{a}$ and $\boldsymbol{\nabla} f$.

5.1.2 Gradient in curvilinear coordinates

The projection of $\boldsymbol{\nabla} f$ onto a covariant base vector, $\mathbf{g}_i$, provides us with the rate of change with respect to the associated contravariant coordinate, x^i,

$$\mathbf{g}_i \cdot \boldsymbol{\nabla} f = \frac{\partial \mathbf{x}}{\partial x^i} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial x^i}. \tag{5.1.4}$$

Consequently,

$$(\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x^i}, \qquad \boldsymbol{\nabla} f = \frac{\partial f}{\partial x^i} \, \mathbf{g}^i. \tag{5.1.5}$$

To signify that the derivatives with respect to contravariant coordinates, x^i, provide us with the covariant components of the gradient, the gradient is sometimes called a covariant vector.

The directional derivative is given by

$$\mathbf{e} \cdot \boldsymbol{\nabla} f = e^i \, \frac{\partial f}{\partial x^i}, \tag{5.1.6}$$

where $\mathbf{e}$ is a unit vector. If $\mathbf{a}$ is an arbitrary vector, then

$$\mathbf{a} \cdot \boldsymbol{\nabla} f = a^i \frac{\partial f}{\partial x^i}, \tag{5.1.7}$$

where summation is implied over the repeated index, i.

Using the rule for raising indices, we find that the contravariant components of the gradient are given by

$$(\boldsymbol{\nabla} f)^i = g^{ik} (\boldsymbol{\nabla} f)_k = g^{ik} \frac{\partial f}{\partial x^k}. \tag{5.1.8}$$

Note that these components are *not* associated with derivatives with respect to covariant coordinates, x^i.

Exercise

5.1.1 Confirm that $\boldsymbol{\nabla} f$ points in the direction of maximum rate of change of f for the function $f = x + y + z$.

5.2 Gradient operator

Referring to equation (5.1.5), we express the gradient operator in the form

$$\boldsymbol{\nabla} = \mathbf{g}^i \frac{\partial}{\partial x^i}, \tag{5.2.1}$$

where summation is implied over the repeated index, i.

5.2.1 Operation on a vector field

For any vector field, $\mathbf{u}$, we may write

$$\boldsymbol{\nabla} \square \mathbf{v} = \mathbf{g}^i \square \frac{\partial \mathbf{v}}{\partial x^i}, \tag{5.2.2}$$

where $\square$ denotes a differential operation such as the inner product $(\cdot)$, the cross product $(\times)$, or the tensor product $(\otimes)$.

Expressing $\mathbf{v}$ in terms of its contravariant or covariant components, we obtain two combinations,

$$\boldsymbol{\nabla} \,\square\, \mathbf{u} = \mathbf{g}^i \,\square\, \frac{\partial}{\partial x^i}(v^j \,\mathbf{g}_j) = \mathbf{g}^i \,\square\, \frac{\partial}{\partial x^i}(v_j \,\mathbf{g}^j), \tag{5.2.3}$$

where summation is implied over the repeated indices, i and j. To derive specific expressions, we expand the derivatives of the products, carry out the $\square$ operations, and express the derivatives of the base vectors, $\mathbf{g}_j$ or $\mathbf{g}_j$, in terms of the Christoffel symbols of the second kind.

Departing from the first expression in (5.2.3), we obtain

$$\boldsymbol{\nabla} \,\square\, \mathbf{u} = \mathbf{g}^i \,\square\, \mathbf{g}_j \,\frac{\partial v^j}{\partial x^i} + v^j \,\mathbf{g}^i \,\square\, \frac{\partial \mathbf{g}_j}{\partial x^i}. \tag{5.2.4}$$

Expressing the derivative $\partial \mathbf{g}_j / \partial x^i$ in terms of the Christoffel symbols of the second kind by way of (4.9.3), repeated below for convenience,

$$\frac{\partial \,\mathbf{g}_j}{\partial x^i} \equiv \Gamma^k_{ij} \,\mathbf{g}_k, \tag{5.2.5}$$

we obtain

$$\boldsymbol{\nabla} \,\square\, \mathbf{v} = \mathbf{g}^i \,\square\, \mathbf{g}_j \,\frac{\partial v^j}{\partial x^i} + v^j \,\Gamma^k_{ji} \,\mathbf{g}^i \,\square\, \mathbf{g}_k, \tag{5.2.6}$$

which can be restated as

$$\boldsymbol{\nabla} \,\square\, \mathbf{v} = \mathbf{g}^i \,\square\, \mathbf{g}_k \left(\frac{\partial v^k}{\partial x^i} + \Gamma^k_{ji} v^j \right). \tag{5.2.7}$$

Renaming the indices, we obtain

$$\boldsymbol{\nabla} \,\square\, \mathbf{v} = \mathbf{g}^j \,\square\, \mathbf{g}_i \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{kj} v^k \right). \tag{5.2.8}$$

Now invoking the definition of the covariant derivative of the contravariant vector components from (4.13.5),

$$v^i_{,j} \equiv \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} \,v^k, \tag{5.2.9}$$

we obtain

$$\boldsymbol{\nabla} \square \mathbf{v} = v^i_{,j}\, \mathbf{g}^j \square \mathbf{g}_i. \tag{5.2.10}$$

Departing from the second expression in (5.2.3) and working in a similar fashion, we obtain

$$\boldsymbol{\nabla} \square \mathbf{v} = v_{i,j}\, \mathbf{g}^j \square \mathbf{g}^i, \tag{5.2.11}$$

where

$$v_{i,j} \equiv \frac{\partial v_i}{\partial x^j} - \Gamma^k_{ji}\, v_k \tag{5.2.12}$$

is the covariant derivative of the covariant vector components defined in (4.13.11).

In Sections 5.3–5.5, we will essentially derive and apply equations (5.2.10) and (5.2.11) for the inner product, the outer product, and the tensor product.

5.2.2 *Operation on a tensor field*

For any tensor field, $\mathbf{T}$, we may write

$$\boldsymbol{\nabla} \square \mathbf{T} = \mathbf{g}^i \square \frac{\partial \mathbf{T}}{\partial x^i}. \tag{5.2.13}$$

Further manipulation involves introducing expansions for $\mathbf{T}$ on the right-hand side and expressing the final result in terms of covariant derivatives of the tensor components defined in Section 4.13. The process will be illustrated in Sections 5.8 and 5.9 for the gradient and the divergence of a tensor field.

Exercise

5.2.1 Derive expression (5.2.11).

5.3 Divergence of a vector field

The divergence of a vector field, $\mathbf{v}$, is a scalar defined by the inner product of the gradient operator $\boldsymbol{\nabla}$ and $\mathbf{v}$, where the inner product is interpreted as an operation, $\boldsymbol{\nabla} \cdot \mathbf{v}$. We find that

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \mathbf{g}^i \cdot \frac{\partial}{\partial x^i}(v^j\, \mathbf{g}_j) = \mathbf{g}^i \cdot \frac{\partial}{\partial x^i}(v_j\, \mathbf{g}^j), \tag{5.3.1}$$

where summation is implied over the repeated indices, i and j.

Expanding the derivative in the first expression in (5.3.1), we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \mathbf{g}^i \cdot \mathbf{g}_j \, \frac{\partial v^j}{\partial x^i} + v^j\, \mathbf{g}^i \cdot \frac{\partial\, \mathbf{g}_j}{\partial x^i}. \tag{5.3.2}$$

Now we recall that $\mathbf{g}^i \cdot \mathbf{g}_j = \delta_{ij}$ and express the last derivative on the right-hand side in terms of the Christoffel symbols of the second kind using the definition (4.9.3), repeated below for convenience,

$$\frac{\partial\, \mathbf{g}_j}{\partial x^i} \equiv \Gamma^k_{ij}\, \mathbf{g}_k, \tag{5.3.3}$$

to obtain

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v^i}{\partial x^i} + \Gamma^k_{ij}\, v^j\, \mathbf{g}^i \cdot \mathbf{g}_k. \tag{5.3.4}$$

Recalling once more that $\mathbf{g}^i \cdot \mathbf{g}_k = \delta_{ik}$, we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v^i}{\partial x^i} + \Gamma^i_{ij}\, v^j, \tag{5.3.5}$$

where summation is implied over the repeated indices, i and j. We see that the divergence of $\mathbf{v}$ is *not* simply the sum of the derivatives of the contravariant components, v^i, with respect to contravariant coordinates, x^i.

In terms of the covariant derivative of the contravariant components introduced in (4.13.5),

$$\boldsymbol{\nabla} \cdot \mathbf{v} = v^i_{,i}, \tag{5.3.6}$$

where summation is implied over the repeated index, i.

5.3.1 Divergence in terms of metric coefficients

Referring to (4.9.18), repeated below for convenience,

$$\Gamma^p_{ij} = \tfrac{1}{2}\, g^{pm} \Big(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \Big), \tag{5.3.7}$$

we find that the coefficient of v^j in the second term on the right-hand side of (5.3.5) is given by

$$\Gamma^i_{ij} = \tfrac{1}{2}\, g^{im} \, \frac{\partial g_{mi}}{\partial x^j}, \tag{5.3.8}$$

where summation is implied over the repeated index, m. In Section 5.6, we will prove that

$$\Gamma^i_{ij} = \frac{1}{\mathcal{J}} \frac{\partial \mathcal{J}}{\partial x^j} = \tfrac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^j}, \tag{5.3.9}$$

as shown in (5.6.22), where $\mathcal{J} = \sqrt{g}$ and g is the determinant of the matrix of covariant coefficients, g. Consequently,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v^i}{\partial x^i} + \frac{1}{\mathcal{J}} \frac{\partial \mathcal{J}}{\partial x^i}\, v^i = \frac{1}{\mathcal{J}} \frac{\partial (\mathcal{J} v^i)}{\partial x^i} \tag{5.3.10}$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v^i}{\partial x^i} + \tfrac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^i}\, v^i. \tag{5.3.11}$$

The second term on the right-hand side of the last equation does not appear when $\mathcal{J}$, and thus g, is spatially uniform, as in the case of oblique rectilinear coordinates.

5.3.2 Laplacian of a scalar field

The Laplacian of a scalar field is the divergence of the gradient.

$$\nabla^2 f \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f \tag{5.3.12}$$

Applying expression (5.3.10) with $\mathbf{v} = \boldsymbol{\nabla} f$ and using expression (5.1.8) for the contravariant components of the gradient,

$$\nabla^2 f = \frac{1}{\mathcal{J}} \frac{\partial}{\partial x^i} \Big(\mathcal{J}\, g^{ki} \frac{\partial f}{\partial x^k} \Big), \tag{5.3.13}$$

as shown previously in (3.5.27). Note that the covariant metric coefficients are involved in this expression. We recall that $\mathcal{J} = \sqrt{g}$, where g is the determinant of the matrix of covariant metric components, $\mathbf{g}$.

Exercise

5.3.1 Show that $\boldsymbol{\nabla}\cdot\mathbf{v} = g^{ij}\, v_{i,j}$, where a comma indicates the covariant derivative.

5.4 *Curl of a vector field*

The curl of a vector field, $\mathbf{v}$, is described by the dual expansion

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{v} = \mathbf{g}^j \times \frac{\partial}{\partial x^j}(v^i\, \mathbf{g}_i) = \mathbf{g}^j \times \frac{\partial}{\partial x^j}(v_i\, \mathbf{g}^i), \tag{5.4.1}$$

where $\times$ denotes the outer (cross) vector product, and summation is implied over the repeated index, i.

Expanding the derivative in the first expression in (5.4.1) into two contributions, we obtain

$$\boldsymbol{\omega} = \frac{\partial v^i}{\partial x^j}\, \mathbf{g}^j \times \mathbf{g}_i + v^i\, \mathbf{g}^j \times \frac{\partial \mathbf{g}_i}{\partial x^j}. \tag{5.4.2}$$

Now we express the last derivative on the right-hand side in terms of the Christoffel symbols of the second kind to obtain

$$\boldsymbol{\omega} = \frac{\partial v^i}{\partial x^j}\, \mathbf{g}^j \times \mathbf{g}_i + \Gamma^k_{ij}\, v^i\, \mathbf{g}^j \times \mathbf{g}_k. \tag{5.4.3}$$

Renaming the repeated index i in the first expression on the right-hand side to k, and grouping the two terms we obtain

$$\boldsymbol{\omega} = v^k_{,j}\, \mathbf{g}^j \times \mathbf{g}_k, \tag{5.4.4}$$

where summation is implied over the repeated indices, j and k and

$$v^k_{,j} = \frac{\partial u^k}{\partial x^j} + u^i \, \Gamma^k_{ij} \tag{5.4.5}$$

is the covariant derivative of the contravariant vector components

Alternatively, we depart from the second expression in (5.4.1) and work in a similar fashion to obtain

$$\boldsymbol{\omega} = v_{k,j} \, \mathbf{g}^j \times \mathbf{g}^k, \tag{5.4.6}$$

where

$$v_{k,j} \equiv \frac{\partial v_k}{\partial x^j} - \Gamma^i_{jk} \, v_i \tag{5.4.7}$$

is the covariant derivative of the covariant vector components defined in (4.13.11).

In three dimensions, we use equations (4.1.14), repeated below for convenience,

$$\mathbf{g}_1 = \mathcal{J} \, \mathbf{g}^2 \times \mathbf{g}^3, \qquad \mathbf{g}_2 = \mathcal{J} \, \mathbf{g}^3 \times \mathbf{g}^1, \qquad \mathbf{g}_3 = \mathcal{J} \, \mathbf{g}^1 \times \mathbf{g}^2, \tag{5.4.8}$$

and obtain

$$\boldsymbol{\omega} = \frac{1}{\mathcal{J}} \Big((v_{3,2} - v_{2,3}) \, \mathbf{g}_1 + (v_{1,3} - v_{3,1}) \, \mathbf{g}_2 + (v_{2,1} - v_{1,2}) \, \mathbf{g}_3 \Big), \tag{5.4.9}$$

which shows that the contravariant components of the vorticity are given by

$$\omega^1 = \frac{1}{\mathcal{J}} \, (v_{3,2} - v_{2,3}), \qquad \omega^2 = \frac{1}{\mathcal{J}} \, (v_{1,3} - v_{3,1}),$$
$$\omega^3 = \frac{1}{\mathcal{J}} \, (v_{2,1} - v_{1,2}). \tag{5.4.10}$$

In compact notation,

$$\omega^i = \frac{1}{\mathcal{J}} \, \epsilon_{ijk} \, v_{k,j}, \qquad \boldsymbol{\omega} = \frac{1}{\mathcal{J}} \, \epsilon_{ijk} \, v_{k,j} \, \mathbf{g}_i, \tag{5.4.11}$$

where ϵ_{ijk} is the Levi–Civita symbol.

Exercise

5.4.1 Confirm that expression (5.4.4) is consistent that in Cartesian coordinates, $\omega_\alpha = \epsilon_{\alpha\beta\gamma}\, \partial u_\gamma / \partial x_\beta$, where Greek indices denote Cartesian coordinates.

5.5 Gradient of a vector field

The gradient of a vector field, $\mathbf{v}$, is a tensor field denoted by

$$\mathbf{L} \equiv \boldsymbol{\nabla} \mathbf{v} \equiv \boldsymbol{\nabla} \otimes \mathbf{v}, \tag{5.5.1}$$

where $\otimes$ is the tensor product. The left projection of a unit vector, $\mathbf{e}$, onto $\mathbf{L}$ is the rate of change of $\mathbf{v}$ with respect to arc length measured in the direction of the unit vector, ℓ,

$$\mathbf{e} \cdot \boldsymbol{\nabla} \mathbf{v} = e_x \frac{\partial \mathbf{v}}{\partial x} + e_y \frac{\partial \mathbf{v}}{\partial y} + e_z \frac{\partial \mathbf{v}}{\partial z} = \frac{\partial \mathbf{v}}{\partial \ell}, \tag{5.5.2}$$

where $e_x^2 + e_y^2 + e_z^2 = 1$.

Using expression (5.2.2), we find that

$$\mathbf{L} = \mathbf{g}^j \otimes \frac{\partial}{\partial x^j}(v^i\, \mathbf{g}_i) = \mathbf{g}^j \otimes \frac{\partial}{\partial x^j}(v_i\, \mathbf{g}^i), \tag{5.5.3}$$

where summation is implied over the repeated indices, i and j.

5.5.1 Contravariant–covariant base

Expanding the derivative with respect to x^j in the first expression of (5.5.3), we obtain

$$\mathbf{L} = \frac{\partial v^i}{\partial x^j}\, \mathbf{g}^j \otimes \mathbf{g}_i + v^i\, \mathbf{g}^j \otimes \frac{\partial \mathbf{g}_i}{\partial x^j}. \tag{5.5.4}$$

Next, we express the last derivative on the right-hand side in terms of the Christoffel symbols, and obtain

$$\mathbf{L} = \frac{\partial v^i}{\partial x^j}\, \mathbf{g}^j \otimes \mathbf{g}_i + \Gamma^k_{ij}\, v^i\, \mathbf{g}^j \otimes \mathbf{g}_k. \tag{5.5.5}$$

Renaming the index i in the first term on the right-hand side to k, and rearranging, we obtain

$$\boldsymbol{\nabla}\mathbf{v} = L_j^{\circ k}\, \mathbf{g}^j \otimes \mathbf{g}_k, \tag{5.5.6}$$

where

$$L_j^{\circ k} = v^k_{,j} \equiv \frac{\partial v^k}{\partial x^j} + \Gamma^k_{ij}\, v^i \tag{5.5.7}$$

is a covariant derivative. We have found that

$$\mathbf{L} \equiv \boldsymbol{\nabla}\mathbf{v} = v^k_{,j}\, \mathbf{g}^j \otimes \mathbf{g}_k, \tag{5.5.8}$$

where summation is implied over the repeated indices, j and k.

5.5.2 Contravariant base

The covariant representation of the gradient is

$$\mathbf{L} = L_{jk}\, \mathbf{g}^j \otimes \mathbf{g}^k, \tag{5.5.9}$$

where

$$L_{jk} = g_{km}\, L_j^{\circ m} = g_{km}\, v^m_{,j} = v_{k,j} \tag{5.5.10}$$

and

$$v_{k,j} = g_{km}\left(\frac{\partial v^m}{\partial x^j} + \Gamma^m_{ij}\, v^i\right) = \frac{\partial v_k}{\partial x^j} - \Gamma^i_{jk}\, v_i \tag{5.5.11}$$

is a covariant derivative, as discussed in Section 4.9.5. We have found that

$$\mathbf{L} \equiv \boldsymbol{\nabla}\mathbf{v} = v_{k,j}\, \mathbf{g}^j \otimes \mathbf{g}^k, \tag{5.5.12}$$

where summation is implied over the repeated indices, j and k.

5.5.3 Divergence of a vector field

We recall that

$$\mathrm{trace}(\mathbf{g}^j \otimes \mathbf{g}_k) = \mathbf{g}^j \cdot \mathbf{g}_k = \delta_{jk}, \tag{5.5.13}$$

and find from (5.5.8) that the divergence of $\mathbf{v}$ is given by

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \mathrm{trace}(\boldsymbol{\nabla}\mathbf{v}) = v^i_{,i}, \tag{5.5.14}$$

where summation is implied over the repeated index, i. This expression is the counterpart of a corresponding expression in Cartesian coordinates indicated by Greek subscripts, $\boldsymbol{\nabla} \cdot \mathbf{v} = \partial v_\alpha / \partial x_\alpha$.

We recall that

$$\mathrm{trace}(\mathbf{g}^j \otimes \mathbf{g}^k) = \mathbf{g}^j \cdot \mathbf{g}^k = g^{jk}, \tag{5.5.15}$$

and find from (5.5.9) that the divergence of $\mathbf{u}$ is also given by

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \mathrm{trace}(\boldsymbol{\nabla}\mathbf{u}) = g^{jk} u_{k,j}, \tag{5.5.16}$$

where summation is implied over the repeated indices, j and k.

5.5.4 Directional derivative

Referring to (5.5.8), we introduce an arbitrary vector, $\mathbf{a}$, we find that

$$\mathbf{a} \cdot \boldsymbol{\nabla}\mathbf{v} = a^j v^k_{,j}\, \mathbf{g}_k. \tag{5.5.17}$$

We may write $\mathbf{a} = |\mathbf{a}|\, \mathbf{e}_a$, where $\mathbf{e}_a$ is the unit vector in the direction of $\mathbf{a}$, and obtain the directional derivative,

$$\mathbf{e}_a \cdot \boldsymbol{\nabla}\mathbf{v} = \frac{1}{|\mathbf{a}|}\, a^j v^k_{,j}\, \mathbf{g}_k. \tag{5.5.18}$$

The contravariant components of $\mathbf{a}$ and $\mathbf{v}$ are involved in this expression.

5.5.5 Symmetric and antisymmetric parts

The gradient of a vector field, $\boldsymbol{\nabla}\mathbf{v}$, can be resolved into a symmetric part given by

$$\mathbf{E} = \tfrac{1}{2}\, v^k_{,j}\, \big(\mathbf{g}^j \otimes \mathbf{g}_k + \mathbf{g}^k \otimes \mathbf{g}_j\big) = \tfrac{1}{2}\, (v^k_{,j} + v^j_{,k})\, \mathbf{g}^j \otimes \mathbf{g}_k \tag{5.5.19}$$

and an antisymmetric part given by

$$\boldsymbol{\Xi} = \tfrac{1}{2}\, v^k_{,j}\, \big(\,\mathbf{g}^j \otimes \mathbf{g}_k - \mathbf{g}^k \otimes \mathbf{g}_j\,\big) = \tfrac{1}{2}\, (v^k_{,j} - v^j_{,k})\, \mathbf{g}^j \otimes \mathbf{g}_k \tag{5.5.20}$$

where $\boldsymbol{\nabla}\mathbf{v} = \mathbf{E} + \boldsymbol{\Xi}$. Alternatively,

$$\mathbf{E} = \tfrac{1}{2}\, v_{k,j}\, \big(\mathbf{g}^j \otimes \mathbf{g}^k + \mathbf{g}^k \otimes \mathbf{g}^j\big) = \tfrac{1}{2}\, (v_{k,j} + v_{j,k})\, \mathbf{g}^j \otimes \mathbf{g}^k \quad (5.5.21)$$

and

$$\boldsymbol{\Xi} = \tfrac{1}{2}\, v_{k,j}\, \big(\mathbf{g}^j \otimes \mathbf{g}^k - \mathbf{g}^k \otimes \mathbf{g}^j\big) = \tfrac{1}{2}\, (v_{k,j} - v_{j,k})\, \mathbf{g}^j \otimes \mathbf{g}^k, \quad (5.5.22)$$

where summation is implied over the repeated indices, j and k.

5.5.6 Curl of a vector field

The alternating tensor, $\boldsymbol{\xi}$, was discussed in Section 4.13, where it was shown that

$$\boldsymbol{\xi} = \frac{1}{\mathcal{J}}\, \epsilon_{pqn}\, \mathbf{g}_{\mathrm{p}} \otimes \mathbf{g}_q \otimes \mathbf{g}_n = \mathcal{J}\, \epsilon_{pqn}\, \mathbf{g}^p \otimes \mathbf{g}^q \otimes \mathbf{g}^n. \qquad (5.5.23)$$

Recalling the representation (5.5.12), we obtain

$$\boldsymbol{\xi} : \mathbf{L} = (\frac{1}{\mathcal{J}}\, \epsilon_{pqn}\, \mathbf{g}_p \otimes \mathbf{g}_q \otimes \mathbf{g}_n) : (v_{k,j}\, \mathbf{g}^j \otimes \mathbf{g}^k). \qquad (5.5.24)$$

Rearranging, we obtain

$$\boldsymbol{\xi} : \mathbf{L} = \frac{1}{\mathcal{J}}\, \epsilon_{pqn}\, v_{k,j}\, (\mathbf{g}_p \otimes \mathbf{g}_q \otimes \mathbf{g}_n) : (\, \mathbf{g}^j \otimes \mathbf{g}^k), \qquad (5.5.25)$$

which can be written as

$$\boldsymbol{\xi} : \mathbf{L} = \frac{1}{\mathcal{J}}\, \epsilon_{pqn}\, v_{k,j}\, \delta_{qj}\, \delta_{nk}\, \mathbf{g}_p = \boldsymbol{\xi} : \mathbf{L} = \frac{1}{\mathcal{J}}\, \epsilon_{pjk}\, v_{k,j}\, \mathbf{g}_p. \quad (5.5.26)$$

which reproduces expression (5.4.11) for the curl,

$$\boldsymbol{\xi} : \boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v}. \qquad (5.5.27)$$

5.5.7 Coordinate transformations

Consider two sets of coordinates, where the second set is indicated with a tilde. The covariant–contravariant components of the gradient of a vector field transform according to the general tensor rule shown in (4.8.20),

$$\widetilde{L}_i^{\circ j} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} L_p^{\circ q}. \tag{5.5.28}$$

Referring to (5.5.7), we obtain

$$\frac{\partial \widetilde{v}^j}{\partial \widetilde{x}^i} + \widetilde{\Gamma}^j_{mi} \widetilde{v}^m = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} \big(\frac{\partial v^q}{\partial x^p} + \Gamma^q_{ip} v^i \big). \tag{5.5.29}$$

Rearranging the right-hand side and invoking the chain rule, we obtain

$$\frac{\partial \widetilde{v}^j}{\partial \widetilde{x}^i} + \widetilde{\Gamma}^j_{mi} \widetilde{v}^m = \frac{\partial \widetilde{x}^j}{\partial x^q} \frac{\partial v^q}{\partial \widetilde{x}^i} + \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} \Gamma^q_{ip} v^i. \tag{5.5.30}$$

Next, we rearrange the first term on the right-hand side and obtain

$$\frac{\partial \widetilde{v}^j}{\partial \widetilde{x}^i} + \widetilde{\Gamma}^j_{mi} \widetilde{v}^m = \frac{\partial}{\partial \widetilde{x}^i} \big(\frac{\partial \widetilde{x}^j}{\partial x^q} v^q \big) - v^q \frac{\partial}{\partial \widetilde{x}^i} \big(\frac{\partial \widetilde{x}^j}{\partial x^q} \big) + \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} \Gamma^q_{ip} v^i. \tag{5.5.31}$$

Recalling from (4.8.15) the vector component transformation rules

$$\widetilde{v}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} v^j, \qquad v^i = \frac{\partial x^i}{\partial \widetilde{x}^j} \widetilde{v}^j, \tag{5.5.32}$$

we find that the first term on the left-hand side of (5.5.31) cancels the first term on the right-hand side. The remaining equation takes the form

$$\widetilde{\Gamma}^j_{mi} \widetilde{v}^m = -\widetilde{v}^m \frac{\partial x^q}{\partial \widetilde{x}^m} \frac{\partial}{\partial \widetilde{x}^i} \big(\frac{\partial \widetilde{x}^j}{\partial x^q} \big) + \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}^j}{\partial x^q} \frac{\partial x^i}{\partial \widetilde{x}^m} \Gamma^q_{ip} \widetilde{v}^m. \tag{5.5.33}$$

Eliminating $\widetilde{v}^m$ from all terms, we recover the transformation rule for the Christofell symbol of the second kind shown in (4.9.28), repeated below for convenience,

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^p} \frac{\partial x^r}{\partial \widetilde{x}^i} \frac{\partial x^s}{\partial \widetilde{x}^j} \Gamma^p_{rs} - \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^r}{\partial \widetilde{x}^j} \frac{\partial^2 \widetilde{x}^k}{\partial x^p \partial x^r}. \tag{5.5.34}$$

5.5.8 Coordinate transformations redux

The covariant components of the gradient of a vector field transform according to the general tensor rule shown in (4.8.19),

$$\widetilde{L}_{ij} = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} L_{pq}. \tag{5.5.35}$$

Referring to (5.5.11), we obtain

$$\frac{\partial \widetilde{v}_j}{\partial \widetilde{x}^i} - \Gamma^m_{ij} \widetilde{v}_m = \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \big(\frac{\partial v_q}{\partial x^p} - \Gamma^m_{pq} v_m\big). \tag{5.5.36}$$

Rearranging the right-hand side, we obtain

$$\frac{\partial \widetilde{v}_j}{\partial \widetilde{x}^i} - \Gamma^m_{ij} \widetilde{v}_m = \frac{\partial x^q}{\partial \widetilde{x}^j} \frac{\partial v_q}{\partial \widetilde{x}^i} - \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \Gamma^m_{pq} v_m. \tag{5.5.37}$$

Rearranging further the first term on the right-hand side, we obtain

$$\frac{\partial \widetilde{v}_j}{\partial \widetilde{x}^i} - \Gamma^m_{ij} \widetilde{v}_m = \frac{\partial}{\partial \widetilde{x}^i} \big(\frac{\partial x^q}{\partial \widetilde{x}^j} v_q \big) - v_q \frac{\partial^2 x^q}{\partial \widetilde{x}^j \partial \widetilde{x}^i} - \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \Gamma^m_{pq} v_m. \tag{5.5.38}$$

We recall from (4.8.16) the vector component transformation rules

$$\widetilde{v}_i = \frac{\partial x^j}{\partial \widetilde{x}^i} v_j, \qquad v_i = \frac{\partial \widetilde{x}^j}{\partial x^i} \widetilde{v}_j, \tag{5.5.39}$$

and find that the first term on the left-hand side of (5.5.38) cancels the first term on the right-hand side. The remaining equation takes the form

$$\Gamma^m_{ij} \widetilde{v}_m = \frac{\partial \widetilde{x}^m}{\partial x^q} \frac{\partial^2 x^q}{\partial \widetilde{x}^j \partial \widetilde{x}^i} \widetilde{v}_m + \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^m}{\partial x^s} \Gamma^s_{pq} \widetilde{v}_m. \tag{5.5.40}$$

Eliminating $\widetilde{v}^m$ from all terms, we recover the transformation rule for the Christofell symbol of the second kind shown in (4.9.22), repeated below for convenience,

$$\widetilde{\Gamma}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^p} \frac{\partial x^r}{\partial \widetilde{x}^i} \frac{\partial x^s}{\partial \widetilde{x}^j} \Gamma^p_{rs} + \frac{\partial^2 x^p}{\partial \widetilde{x}^i \partial \widetilde{x}^j} \frac{\partial \widetilde{x}^k}{\partial x^p}. \tag{5.5.41}$$

Exercise

5.5.1 Explain how (5.5.34) follows from (5.5.33).

5.6 Gradient of a tensor field

The differential operations on vector fields discussed in earlier in this chapter can be extended in a straightforward fashion to two-index tensors fields. The main motivation is the availability of expressions that allow us to state equations of mathematical physics in contravariant or covariant component form.

The gradient of a tensor field, $\mathbf{T}$, is a three-index tensor defined in terms of the tensor product and denoted by

$$\mathbf{N} \equiv \boldsymbol{\nabla}\mathbf{T} \equiv \boldsymbol{\nabla} \otimes \mathbf{T}. \qquad (5.6.1)$$

The left projection of a unit vector, $\mathbf{e}$, onto $\mathbf{N}$ is the rate of change of $\mathbf{N}$ with respect to arc length measured in the direction of the unit vector, ℓ,

$$\mathbf{e} \cdot \boldsymbol{\nabla}\mathbf{N} = e_x \frac{\partial \mathbf{T}}{\partial x} + e_y \frac{\partial \mathbf{T}}{\partial y} + e_z \frac{\partial \mathbf{T}}{\partial z} = \frac{\partial \mathbf{T}}{\partial \ell}, \qquad (5.6.2)$$

where $e_x^2 + e_y^2 + e_z^2 = 1$.

5.6.1 Representation in curvilinear coordinates

Using (5.2.2), we find that

$$\mathbf{N} = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(T^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j) = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(T_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j). \qquad (5.6.3)$$

Two similar expressions can be written involving the mixed components of $\mathbf{T}$,

$$\mathbf{N} = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(T_i^{\circ j}\, \mathbf{g}^i \otimes \mathbf{g}_j) = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(T^i_{\circ j}\, \mathbf{g}_i \otimes \mathbf{g}^j), \qquad (5.6.4)$$

where summation is implied over the three repeated indices, i, j, k.

5.6.2 Covariant derivatives of contravariant components

Expanding the derivative in the first expression in (5.6.3), we obtain

$$\mathbf{N} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\,\mathbf{g}^k \otimes \frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}. \tag{5.6.5}$$

The last two derivatives, $\partial \mathbf{g}_i/\partial x^k$ and $\partial \mathbf{g}_j/\partial x^k$, can be expressed in terms of the Christoffel symbols of the second kind using the definition (4.9.3),

$$\frac{\partial\,\mathbf{g}_i}{\partial x^k} \equiv \Gamma^m_{ik}\,\mathbf{g}_m, \qquad \frac{\partial\,\mathbf{g}_j}{\partial x^k} \equiv \Gamma^m_{jk}\,\mathbf{g}_m. \tag{5.6.6}$$

Substituting these expressions into (5.6.5), we obtain

$$\mathbf{N} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\,\Gamma^m_{ik}\,\mathbf{g}^k \otimes \mathbf{g}_m \otimes \mathbf{g}_j + T^{ij}\,\Gamma^m_{jk}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_m. \tag{5.6.7}$$

Next, we mutually rename the indices i and m in the penultimate term and the indices j and m in the last term on the right-hand side, and obtain

$$\mathbf{N} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j + T^{mj}\,\Gamma^i_{mk}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j + T^{im}\,\Gamma^j_{mk}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j. \tag{5.6.8}$$

We have found that

$$\mathbf{N} = T^{ij}_{,k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j, \tag{5.6.9}$$

where

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\,T^{mj} + \Gamma^j_{mk}\,T^{im} \tag{5.6.10}$$

is the covariant derivative of the contravariant components of $\mathbf{T}$; summation is implied over the repeated index, m.

5.6.3 Covariant derivatives of covariant components

Working in a similar fashion with the second expression in (5.6.3), we find that

$$\mathbf{N} = T_{ij,k}\, \mathbf{g}^k \otimes \mathbf{g}^i \otimes \mathbf{g}^j, \tag{5.6.11}$$

where

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}\, T_{mj} - \Gamma^m_{kj}\, T_{im} \tag{5.6.12}$$

is the covariant derivative of the covariant components.

5.6.4 Covariant derivatives of mixed components

Working in a similar fashion with the first expression in (5.6.4), we find that

$$\mathbf{N} = T^i_{\circ j,k}\, \mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}^j, \tag{5.6.13}$$

where

$$T^i_{\circ j,k} = \frac{\partial T^i_{\circ j}}{\partial x^k} + \Gamma^i_{mk}\, T^m_{\circ j} - \Gamma^m_{jk}\, T^i_{\circ m} \tag{5.6.14}$$

is a covariant derivative of mixed components.

Working in a similar fashion with the second expression in (5.6.4), we find that

$$\mathbf{N} = T^{\circ j}_{i,k}\, \mathbf{g}^k \otimes \mathbf{g}^i \otimes \mathbf{g}_j, \tag{5.6.15}$$

where

$$T^{\circ j}_{i,k} = \frac{\partial T^{\circ j}_i}{\partial x^k} - \Gamma^m_{ik}\, T^{\circ j}_m + \Gamma^j_{km}\, T^{\circ m}_i \tag{5.6.16}$$

is yet another covariant derivative of mixed components.
Note that the sign of the terms involving the Christoffel symbols on the right-hand sides of (5.6.10), (5.6.12), (5.6.14), and (5.6.16) is positive when m appears as a superscript and negative when m appears as a subscript on T.

5.6.5 Ricci's lemma

We recall the following expansion of the metric tensor represented by the identity tensor, $\mathbf{I}$,

$$\mathbf{I} = g_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i = g^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j. \tag{5.6.17}$$

Identifying $\mathbf{T}$ with $\mathbf{I}$, setting $\boldsymbol{\nabla}\mathbf{I} = \mathbf{0}$, and referring to (5.6.9) and (5.6.11), we derive *Ricci's lemma* expressed by

$$g^{ij}_{,k} = 0, \qquad g_{ij,k} = 0. \tag{5.6.18}$$

Substitutions the expressions for the covariant derivatives involved in these equations, we obtain the equations

$$\frac{\partial g_{ij}}{\partial x^k} = g_{mj}\,\Gamma^m_{ik} + g_{im}\,\Gamma^m_{kj},$$
$$\frac{\partial g^{ij}}{\partial x^k} = -g^{mj}\,\Gamma^i_{mk} - g^{im}\,\Gamma^j_{mk}, \tag{5.6.19}$$

expressing the first and second parts of Ricci's lemma.

Since $g^m_{\circ k} = \delta_{mk}$ and $g^i_{\circ m} = \delta_{im}$, we find from (5.6.14) and (5.6.16) that

$$g^i_{\circ j,k} = 0, \qquad g^{\circ j}_{i,k} = 0. \tag{5.6.20}$$

Consequently, equations (5.6.13) and (5.6.15) for $\mathbf{T} = \mathbf{I}$ are identically satisfied.

5.6.6 Derivative of the metric coefficient

From the last entry of Table 4.1.1, we read that

$$\frac{\partial \mathcal{J}}{\partial x^k} = \tfrac{1}{2}\,\mathcal{J}\,g^{ij}\,\frac{\partial g_{ij}}{\partial x^k}. \tag{5.6.21}$$

Substituting the expression for the last derivative $\partial g_{ij}/\partial x^k$ given in (5.6.19), we obtain

$$\frac{\partial \mathcal{J}}{\partial x^k} = \tfrac{1}{2}\,\mathcal{J}\,g^{ij}(g_{mj}\,\Gamma^m_{ik} + g_{im}\,\Gamma^m_{kj}\,). \tag{5.6.22}$$

Simplifying, we obtain

$$\frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial x^k} = \Gamma^m_{mk}, \tag{5.6.23}$$

which provides us with the spatial rate of change of the Jacobian along the kth contravariant coordinate in terms of the Christoffel symbols.

Exercise

5.6.1 Derive (5.6.23) from (5.6.22)

5.7 Divergence of a tensor field

The divergence of a tensor field, $\mathbf{T}$, is a vector field denoted by $\boldsymbol{\psi}$, given by

$$\boldsymbol{\psi} \equiv \boldsymbol{\nabla}\cdot\mathbf{T} = \psi^j \mathbf{g}_j = \psi_j\,\mathbf{g}^j, \tag{5.7.1}$$

where ψ^j are the contravariant components and ψ_j are the covariant components of $\boldsymbol{\psi}$.

Using expression (5.2.1) for the gradient operator, we write

$$\boldsymbol{\psi} = \mathbf{g}^k\cdot\frac{\partial}{\partial x^k}(T^{ij}\,\mathbf{g}_i\otimes\mathbf{g}_j) = \mathbf{g}^k\cdot\frac{\partial}{\partial x^k}(T_{ij}\,\mathbf{g}^i\otimes\mathbf{g}^j). \tag{5.7.2}$$

Two similar expressions can be written involving the mixed components of $\mathbf{T}$,

$$\boldsymbol{\psi} = \mathbf{g}^k\cdot\frac{\partial}{\partial x^k}(T_i^{\circ j}\,\mathbf{g}^i\otimes\mathbf{g}_j) = \mathbf{g}^k\cdot\frac{\partial}{\partial x^k}(T^i_{\circ j}\,\mathbf{g}_i\otimes\mathbf{g}^j), \tag{5.7.3}$$

where summation is implied over the repeated indices, i, j, k.

5.7.1 Contravariant components

Expanding the derivative with respect to x^k in the first expression of (5.7.2), we find that

$$\boldsymbol{\psi} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k\cdot\mathbf{g}_i\otimes\mathbf{g}_j + T^{ij}\,\mathbf{g}^k\cdot\frac{\partial \mathbf{g}_i}{\partial x^k}\otimes\mathbf{g}_j + T^{ij}\,\mathbf{g}^k\cdot\mathbf{g}_i\otimes\frac{\partial \mathbf{g}_j}{\partial x^k}, \tag{5.7.4}$$

where $\mathbf{g}^k \cdot \mathbf{g}_i = \delta_{ki}$ in the first and last terms on the right-hand side. Expressing the derivatives of the covariant base vectors on the right-hand side in terms of the Christoffel symbols of the second kind using the definition (4.9.3), repeated below for convenience,

$$\frac{\partial \mathbf{g}_i}{\partial x^k} \equiv \Gamma^m_{ik}\, \mathbf{g}_m, \qquad \frac{\partial \mathbf{g}_j}{\partial x^k} \equiv \Gamma^m_{jk}\, \mathbf{g}_m, \tag{5.7.5}$$

we obtain

$$\boldsymbol{\psi} = \frac{\partial T^{ij}}{\partial x^i}\, \mathbf{g}_j + T^{ij}\, \Gamma^k_{ik}\, \mathbf{g}_j + T^{ij}\, \Gamma^m_{ji}\, \mathbf{g}_m. \tag{5.7.6}$$

Next, we mutually rename the indices j and m in the last term on the right-hand side, and obtain

$$\boldsymbol{\psi} \equiv \boldsymbol{\nabla} \cdot \mathbf{T} = \frac{\partial T^{ij}}{\partial x^i}\, \mathbf{g}_j + T^{ij}\, \Gamma^k_{ik}\, \mathbf{g}_j + T^{im}\, \Gamma^j_{mi}\, \mathbf{g}_j. \tag{5.7.7}$$

We have found that

$$\boldsymbol{\psi} \equiv \boldsymbol{\nabla} \cdot \mathbf{T} = \psi^j\, \mathbf{g}_j, \tag{5.7.8}$$

where

$$\psi^j = \frac{\partial T^{ij}}{\partial x^i} + \Gamma^k_{mk}\, T^{mj} + \Gamma^j_{mi}\, T^{im} = T^{ij}_{,i} \tag{5.7.9}$$

and the covariant derivative of the contravariant tensor components, indicated by a comma, is defined in (5.6.10) as

$$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\, T^{mj} + \Gamma^j_{mk}\, T^{im}. \tag{5.7.10}$$

The expression in (5.7.9) arising by setting $k = i$.

Working in a similar fashion, with the first expression in (5.7.3), we find the alternative representation

$$\psi^j = T^{\circ j}_{i,k}\, g^{ik}, \tag{5.7.11}$$

where the covariant derivative $T^{\circ j}_{i,k}$ is defined in equation (5.6.16) as

$$T^{\circ j}_{i,k} = \frac{\partial T^{\circ j}_i}{\partial x^k} - \Gamma^m_{ik}\, T^{\circ j}_m + \Gamma^j_{km}\, T^{\circ m}_i. \tag{5.7.12}$$

5.7.2 Covariant components

Working in a similar fashion with the second expressions in (5.7.2) and (5.7.3), we find that the covariant components of $\boldsymbol{\psi}$ are given by

$$\psi_j = T^i_{\circ j,i} = T_{ij,k}\, g^{ik}, \tag{5.7.13}$$

where

$$T^i_{\circ j,k} = \frac{\partial T^i_{\circ j}}{\partial x^k} + \Gamma^i_{mk}\, T^m_{\circ j} - \Gamma^m_{jk}\, T^i_{\circ m} \tag{5.7.14}$$

is a covariant derivative defined in (5.6.14) and

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}\, T_{mj} - \Gamma^m_{kj}\, T_{im} \tag{5.7.15}$$

is another covariant derivative defined in (5.6.12).

5.7.3 Laplacian of a vector field

The Laplacian of a vector field, $\mathbf{u}$, is another vector field, denoted by $\nabla^2\mathbf{u}$, defined as the divergence of the gradient of the field, $\boldsymbol{\nabla} \otimes \mathbf{u}$, that is,

$$\boldsymbol{\psi} \equiv \nabla^2\mathbf{u} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \otimes \mathbf{u}). \tag{5.7.16}$$

Invoking formula (5.5.8) for the gradient of a vector field,

$$\mathbf{T} \equiv \boldsymbol{\nabla}\mathbf{u} = u^j_{,i}\, \mathbf{g}^i \otimes \mathbf{g}_j, \tag{5.7.17}$$

and referring to equation (5.7.11), we set

$$T_i^{\circ j} = u^j_{,i}. \tag{5.7.18}$$

Substituting this expression into (5.7.11), we find that

$$\psi^j = T^{\circ j}_{i,k}\, g^{ik} = (u^j_{,i})_{,k}\, g^{ik} \equiv u^j_{,ik}\, g^{ik}. \tag{5.7.19}$$

In conclusion, we have found that

$$\nabla^2\mathbf{u} = u^j_{,ik}\, g^{ik}\, \mathbf{g}_j, \tag{5.7.20}$$

where summation is implied over the repeated indices i, j, k.

Exercise

5.7.1 Derive the expression shown in (5.7.13).

5.8 Riemann–Christoffel curvature tensor

Following the discussion of Section 5.7, now we identify a tensor, $\mathbf{T}$, with the gradient of a vector field, $\mathbf{u}$,

$$\mathbf{T} = \boldsymbol{\nabla} \otimes \mathbf{u}, \tag{5.8.1}$$

and consider the gradient

$$\mathbf{N} \equiv \boldsymbol{\nabla}\mathbf{T} \equiv \boldsymbol{\nabla} \otimes \mathbf{T} = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{u}. \tag{5.8.2}$$

In Cartesian coordinates indicated by Greek indices, $\mathbf{N}$ is a three-index tensor given by

$$\boldsymbol{\nabla} \otimes \mathbf{T} = \frac{\partial^2 u_\gamma}{\partial x_\alpha \partial x_\beta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma. \tag{5.8.3}$$

The component matrix of second derivatives is symmetric with respect to α and β.

5.8.1 Expression in curvilinear coordinates

We recall from (5.6.11) that

$$\mathbf{N} = T_{ij,k}\, \mathbf{g}^k \otimes \mathbf{g}^i \otimes \mathbf{g}^j, \tag{5.8.4}$$

where

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}\, T_{mj} - \Gamma^m_{kj}\, T_{im} \tag{5.8.5}$$

is the covariant derivative of the covariant components. Substituting $T_{ij} = u_{j,i}$ and recalling that

$$u_{j,i} \equiv \frac{\partial u_j}{\partial x^i} - \Gamma^p_{ij}\, u_p, \tag{5.8.6}$$

we obtain

$$T_{ij,k} = \frac{\partial}{\partial x^k}\Big(\frac{\partial u_j}{\partial x^i} - \Gamma^p_{ij}\, u_p\Big) - \Gamma^m_{ik}\,\big(\frac{\partial u_j}{\partial x^m} - \Gamma^p_{mj}\, u_p\big)$$
$$-\Gamma^m_{kj}\big(\frac{\partial u_m}{\partial x^i} - \Gamma^p_{im}\, u_p\big). \tag{5.8.7}$$

Carrying out the differentiation in the first term on the right-hand side and rearranging, we obtain

$$T_{ij,k} = \big(\,\Gamma^m_{kj}\Gamma^p_{im} - \frac{\partial \Gamma^p_{ij}}{\partial x^k}\,\big)\, u_p \tag{5.8.8}$$
$$+\frac{\partial^2 u_j}{\partial x^k \partial x^i} - \Gamma^p_{ij}\frac{\partial u_p}{\partial x^k} - \Gamma^m_{kj}\frac{\partial u_m}{\partial x^i} - \Gamma^m_{ik}\,\frac{\partial u_j}{\partial x^m} + \Gamma^m_{ik}\,\Gamma^p_{mj}\, u_p.$$

The last five terms on the right-hand side remain unchanged when the indices k and i are mutually switched. Consequently, twice the antisymmetric part of these tensor components with respect to the indices i and k is

$$A_{jik} \equiv T_{ij,k} - T_{kj,i} = \big(\,\Gamma^m_{kj}\Gamma^p_{im} - \frac{\partial \Gamma^p_{ij}}{\partial x^k}\,\big)\, u_p$$
$$-\big(\,\Gamma^m_{ij}\Gamma^p_{km} - \frac{\partial \Gamma^p_{kj}}{\partial x^i}\,\big)\, u_p, \tag{5.8.9}$$

where $A_{jik} = -A_{jki}$. Now we define

$$A_{jik} \equiv \mathcal{R}^p_{\circ jik}\, u_p, \tag{5.8.10}$$

where

$$\mathcal{R}^p_{\circ jik} \equiv \Gamma^p_{im}\Gamma^m_{kj} - \frac{\partial \Gamma^p_{ij}}{\partial x^k} - \Gamma^p_{km}\Gamma^m_{ij} + \frac{\partial \Gamma^p_{kj}}{\partial x^i} \tag{5.8.11}$$

are the components of the Riemann–Christoffel curvature tensor. We have found that

$$\mathbf{A} = \mathbf{u}\cdot(\mathcal{R}^p_{\circ jik}\, \mathbf{g}_p \otimes \mathbf{g}^j \otimes \mathbf{g}^i \otimes \mathbf{g}^k), \tag{5.8.12}$$

which shows that

$$\mathbf{A} = \mathbf{u}\cdot\boldsymbol{\mathcal{R}}, \tag{5.8.13}$$

where

$$\boldsymbol{\mathcal{R}} \equiv \mathcal{R}^p_{\circ jik}\, \mathbf{g}^p \otimes \mathbf{g}^j \otimes \mathbf{g}^i \otimes \mathbf{g}^k \tag{5.8.14}$$

is the Riemann–Christofell curvature tensor. The vanishing of the Riemann–Christoffel curvature tensor for a complete set of curvilinear coordinates, $\boldsymbol{\mathcal{R}} = \mathbf{0}$, to be confirmed later in this section, ensures that $\mathbf{A} = \mathbf{0}$ for any $\mathbf{u}$, as required for the matrix of second derivatives in Cartesian coordinates to be symmetric.

5.8.2 Covariant components

The pure covariant components of the Riemann–Christoffel curvature tensor are given by

$$\mathcal{R}_{ijkp} = g_{iq}\mathcal{R}^q_{\circ jkp}, \tag{5.8.15}$$

where

$$\boldsymbol{\mathcal{R}} = \mathcal{R}_{ijkp}\, \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^p. \tag{5.8.16}$$

We find that

$$\mathcal{R}_{ijkp} = \frac{\partial \Gamma_{i;jp}}{\partial x^k} - \frac{\partial \Gamma_{i;jk}}{\partial x^p} + \Gamma_{q;ip}\,\Gamma^q_{jk} - \Gamma_{q;ik}\,\Gamma^q_{jp}, \tag{5.8.17}$$

where $\Gamma_{q;ip}$ are the Christoffel symbols of the first kind defined and discussed in Section 4.8. The Christoffel symbols of the first kind in equation in equation (5.8.17) can be expressed in terms of the metric coefficients using (4.9.32), yielding

$$\mathcal{R}_{ijkp} = \tfrac{1}{2}\Big(\frac{\partial^2 g_{ip}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^p} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^i \partial x^k}\Big) + g_{nm}\left(\Gamma^n_{jk}\Gamma^m_{ip} - \Gamma^n_{jp}\Gamma^m_{ik}\right). \tag{5.8.18}$$

These expressions reveal that

$$\mathcal{R}_{kimj} = \mathcal{R}_{mjki} = -\mathcal{R}_{ikmj} = -\mathcal{R}_{kijm}. \tag{5.8.19}$$

Moreover,

$$\mathcal{R}_{kimj} + \mathcal{R}_{kmji} + \mathcal{R}_{kjim} = 0. \tag{5.8.20}$$

5.8.3 Vanishing of $\boldsymbol{\mathcal{R}}$

To prove that $\boldsymbol{\mathcal{R}} = \mathbf{0}$, we differentiate the definition of the Christoffel symbols of the second kind given in (4.9.5), repeated below for convenience,

$$\Gamma^k_{ij} = \frac{\partial\, \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k, \tag{5.8.21}$$

and obtain

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2\, \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k + \frac{\partial\, \mathbf{g}_i}{\partial x^j} \cdot \frac{\partial\, \mathbf{g}^k}{\partial x^m}. \tag{5.8.22}$$

Expressing each derivative in the last term on the right-hand side in terms of the Christoffel symbols using (4.9.3) and (4.9.11), we obtain

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2\, \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k - \Gamma^n_{ij}\, \Gamma^k_{pm}\, \mathbf{g}_n \cdot \mathbf{g}^p. \tag{5.8.23}$$

Simplifying the last term, we obtain

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2\, \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k - \Gamma^n_{ij} \Gamma^k_{nm}. \tag{5.8.24}$$

Interchanging the indices j and m, we obtain

$$\frac{\partial \Gamma^k_{im}}{\partial x^j} = \frac{\partial^2\, \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k - \Gamma^n_{im} \Gamma^k_{nj}. \tag{5.8.25}$$

Finally, we subtract the last two equations and derive an identity,

$$\mathcal{R}^k_{\circ imj} \equiv \frac{\partial \Gamma^k_{ji}}{\partial x^m} - \frac{\partial \Gamma^k_{mi}}{\partial x^j} + \Gamma^k_{mn} \Gamma^n_{ji} - \Gamma^k_{jn} \Gamma^n_{mi} = 0, \tag{5.8.26}$$

where $\mathcal{R}^k_{\circ imj}$ are components of the Riemann–Christoffel curvature tensor.

5.8.4 Euclidean v. Riemannian

A Euclidean space is capable of supporting a Cartesian frame where all Christoffel symbols, and thus all components of the Riemann–Christoffel curvature tensor, are identically zero. This means that the components of the Riemann–Christoffel curvature tensor in any coordinate system defined in Euclidean space are also zero, as confirmed by identity (5.8.26).

However, the Riemann–Christoffel curvature tensor is nonzero in a reduced Euclidean dimensional space, such as the surface of a sphere, where the dimension of the base vectors is higher than the number of curvilinear coordinates employed. In this context, the surface of a sphere is a non-Euclidean Riemannian manifold, as discussed in Section 6.4 in the context of surface coordinates.

5.8.5 General relativity

The Ricci curvature tensor is defined as

$$\mathbf{R} = R_{ij}\,\mathbf{g}^i\,\mathbf{g}^j, \tag{5.8.27}$$

where $R_{ij} \equiv \mathcal{R}^k_{\circ ikj}$ are covariant components and summation is implied over the repeated index, k. Einstein's equation of general relativity in the space–time domain reads

$$R_{ij} - \tfrac{1}{2}\,\varrho\,g_{ij} = \frac{8\pi G}{c^4}\,T_{ij}, \tag{5.8.28}$$

where

$$\varrho \equiv \mathrm{trace}(\mathbf{R}) = g^{ij}R_{ij} = R_i^{\circ i} = R^i_{\circ i} \tag{5.8.29}$$

is the scalar curvature, G is the gravitational acceleration, c is the speed of light in vacuum, and T_{ij} are the covariant components of the stress–energy tensor, $\mathbf{T}$. In Section 6.8, we will see that ϱ is proportional to the Gaussian curvature on any surface embedded in three-dimensional space.

Exercise

5.8.1 Derive (5.8.17) from (5.8.26).

5.9 Equations of mathematical physics

A summary of differential operations derived in this chapter for scalars and vectors is shown in Tables 5.9.1. A corresponding summary for tensors is shown in Table 5.9.2. The application of the formulas displayed in these tables to typical equations of mathematical physics will be illustrated in this section. with reference to the continuity equation, the Cauchy equation of motion, and the Navier–Stokes equation.

5.9.1 Continuity equation

The continuity equation ensures that mass is conserved locally and globally in the flow of an incompressible or compressible fluid. The Eulerian form of the continuity equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = 0, \tag{5.9.1}$$

where t stands for time, ρ is the fluid density, and $\mathbf{u}$ is the fluid velocity. In curvilinear coordinates, the continuity equation reads

$$\frac{\partial \rho}{\partial t} + (\rho\, u)^i_{,i} = 0, \tag{5.9.2}$$

where the comma indicates the covariant derivative. Invoking the definition of the covariant derivative, we obtain the explicit form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u^i)}{\partial x^i} + \frac{1}{\mathcal{J}} \frac{\partial \mathcal{J}}{\partial x^i}\, \rho u^i = 0, \tag{5.9.3}$$

where $\mathcal{J} = \sqrt{g}$ and $g = \det(\mathbf{g})$.

5.9.2 Cauchy equation of motion

The Cauchy equation governs the motion of an incompressible or compressible fluid. The Eulerian form of Cauchy's equation reads

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho}\, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{g}, \tag{5.9.4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{g}$ is the gravitational acceleration.

Gradient of a scalar field	$\boldsymbol{\nabla} f = (\boldsymbol{\nabla} f)^i \, \mathbf{g}_i = (\boldsymbol{\nabla} f)_i \, \mathbf{g}^i$
	$(\boldsymbol{\nabla} f)^i = \dfrac{\partial f}{\partial x_i}, \qquad (\boldsymbol{\nabla} f)_i = g_{ij} \, \dfrac{\partial f}{\partial x_j}$
Directional derivatives	$\mathbf{g}_i \cdot \boldsymbol{\nabla} f = \dfrac{\partial f}{\partial x^i}, \quad \mathbf{g}^i \cdot \boldsymbol{\nabla} f = \dfrac{\partial f}{\partial x_i}$
Convective derivative	$\mathbf{v} \cdot \boldsymbol{\nabla} f = v^i f_i = v_i f^i$
Covariant derivative	$u^i_{,j} \equiv \dfrac{\partial u^i}{\partial x^j} + \Gamma^i_{jk} \, u^k$
Covariant derivative	$u_{i,j} \equiv \dfrac{\partial u_i}{\partial x^j} - \Gamma^k_{ji} \, u_k$
Divergence of a vector field	$\boldsymbol{\nabla} \cdot \mathbf{u} = u^i_{,i} = \dfrac{\partial u^i}{\partial x^i} + \dfrac{1}{\mathcal{J}} \, \dfrac{\partial \mathcal{J}}{\partial x^i} \, u^i$
Curl of a vector field	$\boldsymbol{\nabla} \times \mathbf{u} = u^k_{,j} \, \mathbf{g}^j \times \mathbf{g}_k$
Gradient of a vector field	$\boldsymbol{\nabla} \mathbf{u} = u^k_{,j} \, \mathbf{g}^j \otimes \mathbf{g}_k$
Convective derivative of $\mathbf{u}$	$\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{u} = v^j u^k_{,j} \, \mathbf{g}_k$
Laplacian of a scalar field	$\nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \dfrac{\partial f}{\partial x^i \partial x_i} + \Gamma^j_{ij} \, \dfrac{\partial f}{\partial x_i}$
Laplacian of a vector field	$\nabla^2 \mathbf{u} = u^j_{,ik} \, g^{ik} \, \mathbf{g}^j$

TABLE 5.9.1 Summary of expressions for differential operations on scalar and vector fields in curvilinear coordinates, where $\mathcal{J} = \sqrt{g}$ and $g = \det[g_{ij}]$. A comma indicates a covariant derivative.

Covariant derivative	$T^{ij}_{,k} = \dfrac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\, T^{mj} + \Gamma^j_{mk}\, T^{im}$
Covariant derivative	$T_{ij,k} = \dfrac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}\, T_{mj} - \Gamma^m_{kj}\, T_{im}$
Covariant derivative	$T^i_{\circ j,k} = \dfrac{\partial T^i_{\circ j}}{\partial x^k} + \Gamma^i_{mk}\, T^m_{\circ j} - \Gamma^m_{jk}\, T^i_{\circ m}$
Covariant derivative	$T^{\circ j}_{i,k} = \dfrac{\partial T^{\circ j}_i}{\partial x^k} - \Gamma^m_{ik}\, T^{\circ j}_m + \Gamma^j_{km}\, T^{\circ m}_i$
Divergence of a tensor field	$\boldsymbol{\nabla} \cdot \mathbf{T} = T^{ij}_{,i}\, \mathbf{g}_j = T^i_{j,i}\, \mathbf{g}^j$

TABLE 5.9.2 Summary of expressions for differential operations on tensor fields in curvilinear coordinates. A comma indicates a covariant derivative.

Each term in the Cauchy equation of motion is a vector. The ith contravariant component of each term is as follows:

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)^i = \frac{\partial u^i}{\partial t}, \qquad (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u})^i = u^j\, u^i_{,j} \tag{5.9.5}$$

and

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma})^i = \sigma^{ji}_{,j}, \qquad (\mathbf{g})^i = g^i. \tag{5.9.6}$$

Accordingly, the ith contravariant component of the Cauchy equation of motion reads

$$\frac{\partial u^i}{\partial t} + u^j\, u^i_{,j} = \frac{1}{\rho}\, \sigma^{ji}_{,j} + g^i. \tag{5.9.7}$$

The associated covariant components can be deduced using the rule for lowering the indices.

5.9.3 Navier–Stokes equation

The Navier–Stokes equation governs the motion of an incompressible Newtonian fluid. The Eulerian form of the Navier–Stokes equation is

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = -\boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{u} + \rho \, \mathbf{g}, \tag{5.9.8}$$

where p is the pressure and μ is the fluid viscosity.

Each term in the Navier–Stokes equation is a vector. The ith contravariant component of each term is as follows:

$$\left(\frac{\partial \mathbf{u}}{\partial t} \right)^i = \frac{\partial u^i}{\partial t}, \qquad (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u})^i = u^j \, u^i_{,j}, \tag{5.9.9}$$

and

$$(\boldsymbol{\nabla} p)^i = g^{ij} \, \frac{\partial p}{\partial x^j}, \qquad (\nabla^2 \mathbf{u})^i = g^{jk} u^i_{,jk}, \qquad (\mathbf{g})^i = g^i. \tag{5.9.10}$$

Accordingly, the ith contravariant component of the Navier–Stokes equation of motion reads

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \, u^i_{,j} \right) = -g^{ij} \, \frac{\partial p}{\partial x^j} + \mu \, g^{jk} u^i_{,jk} + g^i. \tag{5.9.11}$$

The associated covariant components can be deduced readily by lowering the indices.

Exercises

5.9.1 State the covariant components of each term in the Navier–Stokes equation.

5.9.2 State the contravariant components of each term of an equation of mathematical physics of your choice.

5.10 Moving time derivative

Suppose that a temperature probe is moving with velocity $\boldsymbol{\phi}$ in a temporally and spatially evolving ambient temperature field, $T(t, \mathbf{x})$. The rate of change of the temperature recorded by the probe is described by the *probe time derivative* denoted with a dot,

$$\dot{T}(t, \mathbf{X}(t)), \tag{5.10.1}$$

where $\mathbf{X}(t)$ is the probe position. The probe velocity is

$$\boldsymbol{\phi} \equiv \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}. \tag{5.10.2}$$

Using the chain rule, we obtain

$$\dot{T} = \frac{\partial T}{\partial t} + \frac{\mathrm{d}X_\alpha}{\mathrm{d}t}\frac{\partial T}{\partial x_\alpha} = \frac{\partial T}{\partial t} + \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \cdot \boldsymbol{\nabla} T, \tag{5.10.3}$$

where Greek indices indicate Cartesian coordinates and summation is implied over the repeated index, α. In terms of the probe velocity, we obtain

$$\dot{T} = \frac{\partial T}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} T. \tag{5.10.4}$$

Using the general relation (5.1.7), we find that

$$\boldsymbol{\phi} \cdot \boldsymbol{\nabla} T = \phi_\alpha \frac{\partial T}{\partial x_\alpha} = \phi^i \frac{\partial T}{\partial x^i}, \tag{5.10.5}$$

where Greek indices indicate Cartesian coordinates and summation is implied over the repeated indices, α and i.

Assume that the probe moves in a medium that flows with velocity, $\mathbf{u}(\mathbf{x}, t)$. If the probe velocity is equal to the medium velocity, $\boldsymbol{\phi} = \mathbf{u}$, then the probe time derivative is the material derivative.

5.10.1 Evolution of a scalar recording

Rearranging equation (5.10.4), we obtain the expression

$$\frac{\partial T}{\partial t} = \dot{T} - \boldsymbol{\phi} \cdot \boldsymbol{\nabla} T, \tag{5.10.6}$$

which can be substituted into an evolution equation to replace $\partial T/\partial t$ with the probe time derivative. For example, in the case of the convection equation,

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} T = 0, \tag{5.10.7}$$

we obtain

$$\dot{T} + (\mathbf{v} - \boldsymbol{\phi}) \cdot \boldsymbol{\nabla} T = 0, \tag{5.10.8}$$

where $\mathbf{v}$ is a convection velocity. When $\boldsymbol{\phi} = \mathbf{v}$, we find that $\dot{T} = 0$, which means that the temperature recorded by the probe remains constant in time.

5.10.2 Evolution of a vectorial recording

The rate of change of a vector field, $\mathbf{u}$, recorded by an observer who moves with velocity $\boldsymbol{\phi}$ is given by

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.10.9}$$

where $\boldsymbol{\nabla}\mathbf{u}$ is the gradient of $\mathbf{u}$. In Cartesian coordinates denoted by Greek subscripts, the $\alpha\beta$ component of the tensor $\boldsymbol{\nabla}\mathbf{u}$ is $\partial u_\beta / \partial x_\alpha$, and equation (5.10.9) takes the form

$$\dot{u_\beta} = \frac{\partial u_\beta}{\partial t} + \phi_\alpha \frac{\partial u_\beta}{\partial x_\alpha}. \tag{5.10.10}$$

In general curvilinear coordinates equation (5.10.9) takes the form

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + \phi^j \, u^i_{,j} \, \mathbf{g}_i, \tag{5.10.11}$$

where a comma denotes the covariant derivative.

Identifying the vector field $\mathbf{u}$ with the position, $\mathbf{x}$ or $\mathbf{X}$, we obtain

$$\dot{\mathbf{X}} = \frac{\partial \mathbf{x}}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{x}, \tag{5.10.12}$$

where first term on the right-hand side is zero, $\boldsymbol{\nabla}\mathbf{x} = \mathbf{I}$, and $\mathbf{I}$ is the identity matrix, in agreement with (5.10.2).

5.10.3 Evolution of a vector field

Rearranging equation (5.10.9), we obtain the expression

$$\frac{\partial \mathbf{u}}{\partial t} = \dot{\mathbf{u}} - \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.10.13}$$

which can be substituted into an evolution equation to replace $\partial \mathbf{u}/\partial t$ with the probe time derivative. For example, in the case of the convection equation,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{u} = \mathbf{0}, \tag{5.10.14}$$

we obtain

$$\dot{\mathbf{u}} + (\mathbf{v} - \boldsymbol{\phi}) \cdot \boldsymbol{\nabla} \mathbf{u} = \mathbf{0}, \tag{5.10.15}$$

where $\mathbf{v}$ is a convection velocity. When $\boldsymbol{\phi} = \mathbf{v}$, the vectorial field $\mathbf{u}$ recorded by the probe remains constant in time.

Exercise

5.10.1 The temperature of the atmosphere near the surface of the earth decreases with altitude according to the equation

$$T(z,t) = T_{\text{surface}}(t) - \Gamma(t)\, z, \tag{5.10.16}$$

where $T_{\text{surface}}(t)$ is the surface temperature, $\Gamma(t)$ is the lapse rate, z is the altitude, and t stands for time. A temperature probe is moving vertically with velocity $\phi = \phi_0 \exp(-\nu z)$, where ϕ_0 and ν are two constants. Derive an expression for the derivative $\mathrm{d}T_{\text{probe}}/\mathrm{d}t$ in terms of time, t, *not* involving the altitude.

5.11 Evolving coordinates

A set of contravariant coordinate lines, x^i, and the associated covariant base vectors, $\mathbf{g}_i$, may be evolving in time, t, through space. A fixed point in space, $\mathbf{x}$, corresponds to a time-dependent set of contravariant coordinates described by a function

$$x^i = \mathcal{C}^i(\mathbf{x}, t), \tag{5.11.1}$$

and associated covariant base vectors, as shown in Figure 5.11.1. Time dependence does not appear in the case of stationary coordinates.

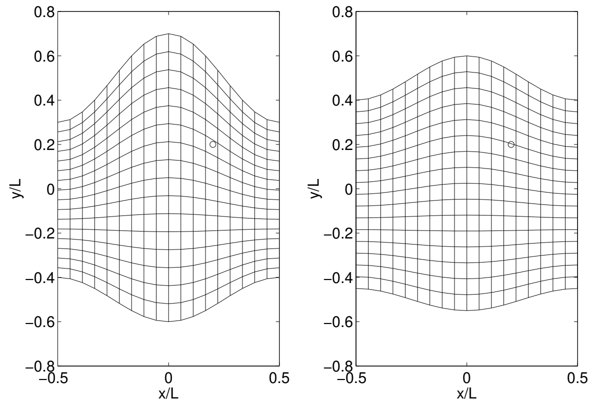

FIGURE 5.11.1 Illustration of evolving curvilinear coordinates shown at two time instants. A certain point in a plane, indicated by a circle, may correspond to a time-dependent set of contravariant coordinates, $x^i(t)$, and associated covariant base vectors, $\mathbf{g}_i(t)$.

Conversely, a certain set of curvilinear coordinates describes a moving point in space whose trajectory is described by an equation

$$\mathbf{x} = \boldsymbol{\mathcal{M}}(x^1, x^2, x^3, t), \tag{5.11.2}$$

where $\boldsymbol{\mathcal{M}}$ is an appropriate function. In the case of stationary coordinates, the point is stationary.

5.11.1 Vector resolution

A vector, $\mathbf{u}$, evaluated at a particular point in space, $\mathbf{x}$, at a certain time instant, t, can be resolved into an evolving set of covariant or contravariant base vectors and associated components as

$$\mathbf{u}(\mathbf{x}, t) = u^i(\mathbf{x}, t)\, \mathbf{g}_i(\mathbf{x}, t) = u_i(\mathbf{x}, t)\, \mathbf{g}^i(\mathbf{x}, t), \tag{5.11.3}$$

where summation is implied over the repeated index, i. The time dependence of the base vectors is due exclusively to the evolution of the coordinates. However, the contravariant or covariant components, $u^i(\mathbf{x}, t)$ and $u_i(\mathbf{x}, t)$, may be evolving even in the case of stationary coordinates.

5.11.2 Evolution of a vector field

In applications, the structure or evolution of a vector field, $\mathbf{u}$, is determined by a governing equation based on a conservation principle or physical law. To express the governing equation in time-dependent coordinates, we employ the notion of the probe time derivative indicated by a dot, as discussed in Section 5.10.

Taking the probe time derivative of the first expansion in (5.11.3), and expanding the derivative using the usual rules of product differentiation, we obtain

$$\dot{\mathbf{u}} = \dot{u}^i \, \mathbf{g}_i + u^i \, \dot{\mathbf{g}}^i, \tag{5.11.4}$$

where

$$\dot{u}^i = \frac{\partial u^i}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} u^i, \qquad \dot{\mathbf{g}}_i = \frac{\partial \mathbf{g}_i}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{g}_i, \tag{5.11.5}$$

and $\boldsymbol{\phi}$ is the probe velocity. Combining equation (5.11.4) with equation (5.10.9), repeated below for convenience,

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.11.6}$$

and rearranging, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \dot{u}^i \, \mathbf{g}_i + u^i \, \dot{\mathbf{g}}_i - \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.11.7}$$

which can be substituted into an evolution equation for $\mathbf{u}$ to express $\partial \mathbf{u}/\partial t$ in terms of probe time derivatives.

5.11.3 Resolution of the probe velocity

We may introduce the contravariant and covariant components of $\boldsymbol{\phi}$ and write the expansions

$$\begin{aligned} \boldsymbol{\phi}(\mathbf{X}(t), t) &= \phi^i(\mathbf{X}(t), t) \, \mathbf{g}_i(\mathbf{X}(t), t) \\ &= \phi_i(\mathbf{X}(t), t) \, \mathbf{g}^i(\mathbf{X}(t), t), \end{aligned} \tag{5.11.8}$$

where $\mathbf{X}(t)$ is the probe position. Substituting the first expression into the last term on the right-hand side of (5.11.7), we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \dot{u}^i \, \mathbf{g}_i + u^i \, \dot{\mathbf{g}}_i - \phi^i \, \mathbf{g}_i \cdot \boldsymbol{\nabla} \mathbf{u}. \tag{5.11.9}$$

Using expansion (5.5.8) for the gradient $\nabla \mathbf{u}$, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \dot{u}^i\, \mathbf{g}_i + u^i\, \dot{\mathbf{g}}_i - \phi^j u^i_{,j}\, \mathbf{g}_i, \tag{5.11.10}$$

where

$$u^i_{,j} \equiv \frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj}\, u^k \tag{5.11.11}$$

is a covariant derivative. The right-hand side of (5.11.10) can be substituted into an evolution equation involving $\partial \mathbf{u}/\partial t$.

5.11.4 One dimension

In the case of one dimension over the x axis, we introduce a contravariant coordinate, x^1, and write

$$\mathbf{g}_1 = g_1\, \mathbf{e}_x, \qquad \mathbf{g}^1 = g^1\, \mathbf{e}_x, \qquad g_1 = \frac{\partial x}{\partial x^1}, \qquad g^1 = \frac{\partial x^1}{\partial x}, \tag{5.11.12}$$

where $\mathbf{e}_x$ is the unit vector along the x axis. The only non-zero Christofell symbols of the second kind is

$$\Gamma^1_{11} = \frac{\partial g_1}{\partial x}. \tag{5.11.13}$$

Next, we expand

$$\boldsymbol{\phi} = \phi\, \mathbf{e}_x = \phi^1\, \mathbf{g}_1 = \phi^1 g_1\, \mathbf{e}_x, \tag{5.11.14}$$

and

$$\mathbf{u} = u\, \mathbf{e}_x = u^1\, \mathbf{g}_1 = u^1\, g_1\, \mathbf{e}_x, \tag{5.11.15}$$

where u^1 and ϕ^1 are contravariant components. The probe velocity component, ϕ^1 can be arbitrarily prescribed. The covariant derivative of u^1 is given by

$$u^1_{,1} = \frac{\partial u^1}{\partial x^1} + \frac{\partial g_1}{\partial x}\, u^1. \tag{5.11.16}$$

Manipulating the derivatives, we obtain

$$u^1_{,1} = \frac{\partial x}{\partial x^1} \frac{\partial u^1}{\partial x} + \frac{\partial g_1}{\partial x}\, u^1 = g_1\, \frac{\partial u^1}{\partial x} + \frac{\partial g_1}{\partial x}\, u^1 \tag{5.11.17}$$

and then

$$u^1_{,1} = \frac{\partial(u^1 g_1)}{\partial x} = \frac{\partial u}{\partial x}. \tag{5.11.18}$$

Substituting these expressions into equation (5.11.7) and eliminating $\mathbf{e}_x$ from both sides, we obtain

$$\frac{\partial u}{\partial t} = \dot{u}^1 \, g_1 + u^1 \, \dot{g}_1 - \phi \, \frac{\partial u}{\partial x}, \tag{5.11.19}$$

which can be restated as

$$\frac{\partial u}{\partial t} = \big(\, \dot{u}^1 + u^1 \, \dot{\ln} g_1 - \phi^1 \, \frac{\partial u}{\partial x} \, \big) \, g_1. \tag{5.11.20}$$

These equations also arise directly from (5.11.10).

Exercise

5.11.1 Confirm (5.11.19) for the case of a uniform distribution where v is a time-dependent uniform field.

5.12 Moving coordinates

With reference to the evolving coordinates discussed in Section 5.11, now we assume that probes move to find themselves at a position corresponding to fixed contravariant coordinates, x^i. The probe time derivative of any appropriate field function, ψ, is the time derivative under constant x^i,

$$\dot{\psi} = \Big(\frac{\partial \psi}{\partial t} \Big)_{x^1, x^2, x^3}. \tag{5.12.1}$$

The probe velocity is given by the time derivative

$$\boldsymbol{\phi} = \Big(\frac{\partial \mathbf{X}}{\partial t} \Big)_{x^1, x^2, x^3}, \tag{5.12.2}$$

where $\mathbf{X}$ is the probe position. By definition,

$$\dot{X}^i = 0 \tag{5.12.3}$$

for $i = 1, 2, 3$.

5.12.1 Distributed probes

Probes could be distributed throughout the entire space so that the probe velocity may be considered a function of arbitrary position in space and time,

$$\boldsymbol{\phi}(\mathbf{x}, t), \tag{5.12.4}$$

where position could be determined by instantaneous contravariant coordinates.

In Section 5.10, we derived the evolution equation (5.10.6) for a distributed scalar field, T,

$$\frac{\partial T}{\partial t} = \dot{T} - \boldsymbol{\phi} \cdot \boldsymbol{\nabla} T. \tag{5.12.5}$$

Applying this equation for $T = X^i$, and recalling that the contravariant base vectors are given by $\mathbf{g}^i = \boldsymbol{\nabla} x^i$, as shown in (4.2.6), we obtain

$$\frac{\partial X^i}{\partial t} = -\boldsymbol{\phi} \cdot \boldsymbol{\nabla} x^i = -\boldsymbol{\phi}^j \, \mathbf{g}_j \, \cdot \boldsymbol{\nabla} x^i \tag{5.12.6}$$

and then

$$\frac{\partial X^i}{\partial t} = -\boldsymbol{\phi}^j \, \mathbf{g}_j \, \cdot \mathbf{g}^i = -\boldsymbol{\phi}^j \, \delta_{ij} = -\boldsymbol{\phi}^i, \tag{5.12.7}$$

yielding

$$\boldsymbol{\phi}^i = -\frac{\partial X^i}{\partial t}, \tag{5.12.8}$$

where the partial derivative on the right-hand side is taken under constant position, $\mathbf{x}$.

Now recalling that the covariant base vectors at a certain time are defined as $\mathbf{g}_i \equiv \partial \mathbf{X} / \partial x^i$, we find that

$$\left(\frac{\partial \, \mathbf{g}_i}{\partial t} \right)_{x^1, x^2, x^3} = \frac{\partial^2 \mathbf{X}}{\partial x^i \partial t} = \frac{\partial}{\partial x^i} \left(\frac{\partial \mathbf{X}}{\partial t} \right) = \frac{\partial \boldsymbol{\phi}}{\partial x^i}. \tag{5.12.9}$$

Using (4.13.4) to write

$$\frac{\partial \boldsymbol{\phi}}{\partial x^i} = \phi^j_{,i}\, \mathbf{g}_j, \tag{5.12.10}$$

where

$$\phi^j_{,i} \equiv \frac{\partial \phi^j}{\partial x^i} + \Gamma^j_{ik}\, u^k \tag{5.12.11}$$

is the covariant derivative, we obtain the expression

$$\Big(\frac{\partial \mathbf{g}_i}{\partial t}\Big)_{x^1,x^2,x^3} = \phi^j_{,i}\, \mathbf{g}_j, \tag{5.12.12}$$

where summation is implied over the repeated index, j.

5.12.2 *Evolution of a vectorial recording*

Equation (5.10.9) for a vector field, $\mathbf{u}$, becomes

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)_{x^1,x^2,x^3} = \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.12.13}$$

and equation (5.11.4) becomes

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)_{x^1,x^2,x^3} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + u^i \Big(\frac{\partial \mathbf{g}_i}{\partial t}\Big)_{x^1,x^2,x^3}. \tag{5.12.14}$$

Substituting expression (5.12.12) into the last term on the right-hand side of (5.12.14), we obtain

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)_{x^1,x^2,x^3} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + u^i \phi^j_{,i}\, \mathbf{g}_j, \tag{5.12.15}$$

which can be restated as

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)_{x^1,x^2,x^3} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\phi}, \tag{5.12.16}$$

where $\boldsymbol{\nabla}\boldsymbol{\phi}$ is the matrix gradient of $\boldsymbol{\phi}$.

Now setting the right-hand side of (5.12.13) equal to the right-hand side of (5.12.16), and rearranging, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\phi} - \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{5.12.17}$$

which can be restated as

$$\frac{\partial \mathbf{u}}{\partial t} = \Big(\frac{\partial \mathbf{u}}{\partial t}\Big)^i \mathbf{g}_i, \tag{5.12.18}$$

where

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)^i = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} + u^j\, \phi^i_{,j} - \phi^j\, u^i_{,j} \tag{5.12.19}$$

is the ith contravariant component of the Eulerian time derivative, $\partial \mathbf{u}/\partial t$. The expressions given in (5.12.17) and (5.12.18) allow us to express the ith contravariant component of the Eulerian derivative on the left-hand side in terms of a coordinate probe time derivative.

To derive a mnemonic rule, we rearrange equations (5.12.17) and (5.12.19) to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\phi} \cdot \boldsymbol{\nabla} \mathbf{u} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\phi} \tag{5.12.20}$$

and

$$\Big(\frac{\partial \mathbf{u}}{\partial t}\Big)^i + \phi^j\, u^i_{,j} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} + u^j\, \phi^i_{,j}, \tag{5.12.21}$$

subject to (5.12.2).

5.12.3 Cauchy equation of motion

As an application, we consider the Cauchy equation of motion discussed in Section 5.1,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho}\, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{g}. \tag{5.12.22}$$

Using expression (5.12.17) for the time derivative, we find that, in a moving coordinate system, this equation takes the form

$$\Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\phi} + (\mathbf{u} - \boldsymbol{\phi}) \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho}\, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{g}, \tag{5.12.23}$$

subject to (5.12.2). The ith contravariant component of this equation reads

$$\left(\frac{\partial u^i}{\partial t}\right)_{x^1,x^2,x^3} + u^j\,\phi^i_{,j} + (u^j - \phi^j)\,u^i_{,j} = \frac{1}{\rho}\,\sigma^{ji}_{,j} + g^i, \tag{5.12.24}$$

where a comma denotes a covariant derivative,

5.12.4 One dimension

In the case of one dimension over the x axis,

$$\phi = \left(\frac{\partial x}{\partial t}\right)_{x^1}, \qquad \left(\frac{\partial g_1}{\partial t}\right)_{x^1} = \frac{\partial \phi}{\partial x^1}, \tag{5.12.25}$$

and ϕ is a specified velocity along the x axis. Equation (5.11.19) becomes

$$\left(\frac{\partial u}{\partial t}\right)_x = \left(\frac{\partial u^1}{\partial t}\right)_{x^1} g_1 + u\,\frac{\partial \phi}{\partial x} - \phi\,\frac{\partial u}{\partial x}. \tag{5.12.26}$$

Consider a one-dimensional distribution, $\Phi(x,t)$, propagating along the x axis with phase velocity c, so that

$$u = \Phi(x - c\,t). \tag{5.12.27}$$

Substituting this expression into (5.12.26), we obtain

$$-c\,\Phi' = \left(\frac{\partial u^1}{\partial t}\right)_{x^1} g_1 + \Phi\,\frac{\partial \phi}{\partial x} - \phi\,\Phi', \tag{5.12.28}$$

where a prime denotes a derivative with respect to $w \equiv x - ct$, Rearranging, we obtain

$$\left(\frac{\partial u^1}{\partial t}\right)_{x^1} g_1 = (\phi - c)\,\Phi' - \Phi\,\frac{\partial \phi}{\partial x}. \tag{5.12.29}$$

We see that it is beneficial to set $\phi = c$ so that the right-hand side is zero.

5.12.5 Translating and rotating coordinates

In the case of translating and rotating coordinates,

$$\boldsymbol{\phi}(\mathbf{x},t) = \mathbf{U}(t) + \boldsymbol{\Omega}(t) \times \big(\mathbf{x} - \mathbf{x}_0(t)\big), \tag{5.12.30}$$

where $\mathbf{U}$ is the coordinate system velocity of translation and $\boldsymbol{\Omega}$ is the angular velocity of rotation about the instantaneous origin of the moving coordinates, $\mathbf{x}_0$. In the moving system, the vector $\mathbf{u}$ appears as $\mathbf{v}$, where

$$\mathbf{u} = \mathbf{v} + \boldsymbol{\phi}. \tag{5.12.31}$$

The left-hand side of the equation of motion (5.12.23) becomes

$$\Big(\frac{\partial(v^i + \phi^i)}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + (\mathbf{v} + \boldsymbol{\phi}) \cdot \boldsymbol{\nabla}\boldsymbol{\phi} + \mathbf{v} \cdot \boldsymbol{\nabla}(\mathbf{v} + \boldsymbol{\phi}), \tag{5.12.32}$$

which can be simplified to

$$\Big(\frac{\partial v^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} + \mathbf{A}, \tag{5.12.33}$$

where

$$\mathbf{A} = \frac{\partial \boldsymbol{\phi}}{\partial t} + (2\mathbf{v} + \boldsymbol{\phi}) \cdot \boldsymbol{\nabla}\boldsymbol{\phi}. \tag{5.12.34}$$

Substituting expression (5.12.30) for $\boldsymbol{\phi}$, we find that

$$\mathbf{A} = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \big(\mathbf{x} - \mathbf{x}_0\big) - \boldsymbol{\Omega} \times \frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}t} \\ + \boldsymbol{\Omega} \times (2\mathbf{v} + \mathbf{U} + \boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_0)). \tag{5.12.35}$$

Setting $\mathrm{d}\mathbf{x}_0/\mathrm{d}t = \mathbf{U}$ and simplifying, we obtain

$$\mathbf{A} = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \hat{\mathbf{x}} + 2\boldsymbol{\Omega} \times \mathbf{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}}), \tag{5.12.36}$$

where $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$. The negative of each term multiplied by the density represent the linear acceleration force, $-\rho\,\mathrm{d}\mathbf{U}/\mathrm{d}t$, the Coriolis force, $-2\,\rho\,\boldsymbol{\Omega} \times \mathbf{v}$, the centrifugal force, $-\rho\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}})$, and the angular acceleration force, $-\rho\,(\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t) \times \hat{\mathbf{x}}$.

5.12.6 Translating field

As an example, we consider the evolution of a uniform vector field, $\mathbf{u} = \mathbf{U}(t)$, where $\mathbf{U}(t)$ is a function of time. Equation (5.12.17) reduces to

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \mathbf{U} \cdot \boldsymbol{\nabla}\boldsymbol{\phi}. \tag{5.12.37}$$

As expected, the second term on the right-hand side is absent when the curvilinear coordinates move as a rigid body, that is, $\boldsymbol{\phi}$ is constant. When $\boldsymbol{\phi}$ expresses rigid-body rotation about a point $\mathbf{x}_\mathrm{R}$ with angular velocity $\boldsymbol{\Omega}$, we set $\boldsymbol{\phi} = \boldsymbol{\Omega} \times \mathbf{x}$ and obtain

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i + \boldsymbol{\Omega} \times \mathbf{U}. \tag{5.12.38}$$

Exercise

5.12.1 Simplify equation (5.12.17) for a flow expressing rigid-body rotation.

5.13 Convected coordinates

Differential equations often involve a velocity field, $\mathbf{u}$. When a set of curvilinear coordinates are convected with this velocity field,

$$\boldsymbol{\phi} = \mathbf{u}, \tag{5.13.1}$$

the differential equations are considerably simplified. Referring to equation (5.12.17), we set $\phi = \mathbf{u}$ and derive the simplified expression

$$\frac{\partial \mathbf{u}}{\partial t} = \Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} \mathbf{g}_i, \tag{5.13.2}$$

where the convected covariant base vectors, $\mathbf{g}_i$, generally depend on position, $\mathbf{x}$, and time, t.

5.13.1 Cauchy equation of motion

Using (5.13.2), we find that, in convected coordinates, the ith contravariant component of Cauchy's equation of motion (5.12.24) takes the form

$$\Big(\frac{\partial u^i}{\partial t}\Big)_{x^1,x^2,x^3} + u^j\, u^i_{,j} = \frac{1}{\rho}\,\sigma^{ji}_{,j} + g^i, \qquad (5.13.3)$$

where a comma indicates a covariant derivative. In fact, this equation is precisely the same as that in stationary coordinates, as shown in (5.9.7).

5.13.2 Point source in oscillatory streaming flow

Consider the temperature field generated by a three-dimensional point source of heat located at the origin, in the presence of an oscillatory streaming flow. The Cartesian velocity components along the x, y, and z axes are given by

$$u_x = U\,\sin(\omega t + \phi), \qquad u_y = 0, \qquad u_z = 0, \qquad (5.13.4)$$

where U is the amplitude of the velocity, ω is the angular frequency, and ϕ is the phase shift. The source is activated impulsively at the origin of time, $t = 0$, by sending an electrical signal through a wire.

The induced temperature field, $\mathcal{G}$, identified as a Green's function, satisfies the three-dimensional singularly forced convection–diffusion equation

$$\frac{\partial \mathcal{G}}{\partial t} + U\,\sin(\omega t + \phi)\,\frac{\partial \mathcal{G}}{\partial x} = \kappa\,(\,\frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial y^2} + \frac{\partial^2 \mathcal{G}}{\partial z^2}\,) + \delta(x)\,\delta(y)\,\delta(z)\,\delta(t), \qquad (5.13.5)$$

where δ is the one-dimensional Dirac delta function, and κ is the medium thermal diffusivity.

To find the solution, we introduce convected Cartesian coordinates, ξ, η, and ζ, defined by

$$x = \xi - \frac{U}{\omega}\,\cos(\omega t + \phi), \qquad y = \eta, \qquad z = \zeta. \qquad (5.13.6)$$

The inverse relations are

$$\xi = x + \frac{U}{\omega}\,\cos(\omega t + \phi), \qquad \eta = y, \qquad \zeta = z. \qquad (5.13.7)$$

Since

$$\boldsymbol{\phi} = \Big(\frac{\partial \mathbf{x}}{\partial t}\Big)_{\xi,\eta,\zeta} = \mathbf{u}, \qquad (5.13.8)$$

the convected coordinates follow material point particles moving under the influence of the oscillatory flow.

The Green's function can be regarded as a function of ξ, η, ζ, and time, t,

$$\mathcal{G}(x, y, z, t) = \mathcal{Q}(\xi, \eta, \zeta, t). \qquad (5.13.9)$$

Using the chain rule, we find that

$$\frac{\partial \mathcal{G}}{\partial t} = \frac{\partial \mathcal{Q}}{\partial t} + \frac{\partial \mathcal{Q}}{\partial \xi}\,\frac{\partial x^1}{\partial t} + \frac{\partial \mathcal{Q}}{\partial \eta}\,\frac{\partial x^2}{\partial t} + \frac{\partial \mathcal{Q}}{\partial \zeta}\,\frac{\partial x^3}{\partial t}, \qquad (5.13.10)$$

and then

$$\frac{\partial \mathcal{G}}{\partial t} = \frac{\partial \mathcal{Q}}{\partial t} - U\sin(\omega t + \phi)\,\frac{\partial \mathcal{Q}}{\partial \xi}. \qquad (5.13.11)$$

Similarly, we find that

$$\frac{\partial \mathcal{G}}{\partial x} = \frac{\partial \mathcal{Q}}{\partial \xi}. \qquad (5.13.12)$$

Substituting these expressions into (5.13.5) and using expression (3.7.5) for the Laplacian, we derive a diffusion equation,

$$\frac{1}{\kappa}\,\frac{\partial \mathcal{Q}}{\partial t} = \frac{\partial^2 \mathcal{Q}}{\partial \xi^2} + \frac{\partial^2 \mathcal{Q}}{\partial \eta^2} + \frac{\partial^2 \mathcal{Q}}{\partial \zeta^2} + \frac{1}{\kappa}\,\delta(\xi)\,\delta(\eta)\,\delta(\zeta)\,\delta(t). \qquad (5.13.13)$$

The well-known solution of this equation is

$$\mathcal{Q}(\xi, \eta, \zeta, t) = \frac{1}{(4\pi\kappa t)^{3/2}}\,\exp\big(-\frac{\xi^2 + \eta^2 + \zeta^2}{4\kappa t}\big). \qquad (5.13.14)$$

Substituting the expressions for ξ, η, and ζ in terms of x, y, z and t, we obtain the Green's function

$$\mathcal{G}(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \exp\big(- \frac{(x + \delta\cos(\omega t + \phi))^2 + y^2 + z^2}{4\kappa t} \big), \tag{5.13.15}$$

where $\delta = U/\omega$ is the amplitude of the displacement.

To obtain an expression for steady streaming flow with velocity U, we set $\phi = \frac{1}{2}\pi$ and take the limit as ω tends to zero to find that $\delta\cos(\omega t + \phi) \to -Ut$.

Exercise

5.13.1 Confirm the limit discussed in the last paragraph of this section.

5.14 Green's functions

The Green's function of the convection–diffusion equation in d dimensions, denoted by $\mathcal{G}$, satisfies the equation

$$\frac{\partial \mathcal{G}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathcal{G} = \boldsymbol{\nabla} \cdot (\mathbf{D} \cdot \boldsymbol{\nabla}\mathcal{G}) + \delta_d(\mathbf{x})\,\delta(t), \tag{5.14.1}$$

where $\mathbf{u}$ is a specified velocity field, $\mathbf{D}$ is a symmetric diffusivity tensor, δ_d is the d-dimensional Dirac delta function, and δ is the one-dimensional Dirac delta function. The Green's function has units of $1/L^d$, where L is a specified length.

For an isotropic medium, $\mathbf{D} = \mathcal{D}\,\mathbf{I}$, where $\mathcal{D}$ is the scalar diffusivity and $\mathbf{I}$ is the identity matrix.

5.14.1 Convected coordinates

It is beneficial to introduce convected contravariant coordinates, denoted as ξ_i. The Green's function can be regarded as a function of ξ_1, ξ_2, ξ_3, and t,

$$\mathcal{G}(x, y, z, t) = \mathcal{Q}(\xi_1, \xi_2, \xi_3, t). \tag{5.14.2}$$

The transformation is designed so that equation (5.14.1) becomes

$$\frac{\partial \mathcal{Q}}{\partial t} = \widehat{\boldsymbol{\nabla}} \cdot (\widehat{\mathbf{D}} \cdot \widehat{\boldsymbol{\nabla}} \mathcal{Q}) + \delta_d(\boldsymbol{\xi})\, \delta(t), \tag{5.14.3}$$

where the gradient $\widehat{\boldsymbol{\nabla}}$ operates with respect to $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$, regarded as Cartesian coordinates in parameter space,

$$\widehat{\mathbf{D}} = \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-\mathrm{T}}, \qquad \mathbf{F}^{\mathrm{T}} = \widehat{\boldsymbol{\nabla}} \mathbf{x}, \qquad \mathbf{F}^{-\mathrm{T}} = \boldsymbol{\nabla} \boldsymbol{\xi}, \tag{5.14.4}$$

the superscript -1 denotes the inverse, and the superscript $-\mathrm{T}$ denotes the inverse of the transport. The deformation gradient, $\mathbf{F}$, is defined such that

$$F_{ij} = \frac{\partial x_i}{\partial \xi_j}, \tag{5.14.5}$$

where $x_1 = x$, $x_2 = y$, and $x_2 = z$.

Since the coordinates are convected,

$$\mathbf{u} = \left(\frac{\partial \mathbf{x}}{\partial t} \right)_{\boldsymbol{\xi}} \tag{5.14.6}$$

and

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_j}{\partial \xi_m} \frac{\partial \xi_m}{\partial x_i} = \frac{\partial}{\partial t} \Big(\frac{\partial x_j}{\partial \xi_m} \Big)_{\boldsymbol{\xi}} F^{-\mathrm{T}}_{im} = F^{-\mathrm{T}}_{im} \left(\frac{\partial F_{jm}}{\partial t} \right)_{\boldsymbol{\xi}}, \tag{5.14.7}$$

which shows that

$$\boldsymbol{\nabla} \mathbf{u} = \mathbf{F}^{-\mathrm{T}} \cdot \left(\frac{\partial \mathbf{F}^{\mathrm{T}}}{\partial t} \right)_{\boldsymbol{\xi}}, \qquad (\boldsymbol{\nabla} \mathbf{u})^{\mathrm{T}} = \left(\frac{\partial \mathbf{F}}{\partial t} \right)_{\boldsymbol{\xi}} \cdot \mathbf{F}^{-1}. \tag{5.14.8}$$

5.14.2 Homogeneous deformation

In the case of a homogeneous deformation, the deformation gradient $\mathbf{F}$ depends on time alone such that

$$\mathbf{x} = \mathbf{F}(t) \cdot \boldsymbol{\xi}, \tag{5.14.9}$$

in agreement with (5.14.5). It can be shown that the solution of (5.14.3) is given by

$$\mathcal{Q}(\boldsymbol{\xi}, t) = \frac{1}{2^d \pi^d \det(\mathbf{L})} \exp\big(-\tfrac{1}{2}\, \boldsymbol{\xi} \cdot \mathbf{L}^{-1} \cdot \boldsymbol{\xi} \big), \tag{5.14.10}$$

where

$$\mathbf{L}(t) = 2 \int_0^t \widehat{\mathbf{D}}(t')\, \mathrm{d}t'. \tag{5.14.11}$$

Substituting

$$\boldsymbol{\xi} = \mathbf{F}^{-1} \cdot \mathbf{x}, \tag{5.14.12}$$

we obtain the final expression

$$\mathcal{G}(\mathbf{x}, t) = \frac{1}{2^d \pi^d \det(\mathbf{L})} \exp\big(-\tfrac{1}{2}\, \mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{L}^{-1} \cdot \mathbf{F}^{-1} \cdot \mathbf{x} \big). \tag{5.14.13}$$

Specific cases will be discussed in the remainder of this chapter.

Exercise

5.14.1 Simplify (5.14.13) for a time-independent deformation gradient, $\mathbf{F}$.

5.15 Point source in simple shear flow

Consider the temperature field generated by a two-dimensional point source of heat or mass located at the origin of the xy plane, in the presence of a simple shear flow flow. The velocity components along the x and y axes are given by

$$u_x = a\, y, \qquad u_y = 0, \tag{5.15.1}$$

where a is the shear rate. The source is activated impulsively at the origin of time, $t = 0$, by sending an electrical signal through a wire.

The induced temperature field, $\mathcal{G}(x, y, t)$, is the Green's function of the associated singularly forced two-dimensional convection–diffusion equation,

$$\frac{\partial \mathcal{G}}{\partial t} + a y \, \frac{\partial \mathcal{G}}{\partial x} = \mathcal{D} \, (\frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial y^2}) + \delta(x) \, \delta(y) \, \delta(t), \qquad (5.15.2)$$

where δ is the one-dimensional Dirac delta function and $\mathcal{D}$ is the medium thermal diffusivity.

5.15.1 *Convected oblique rectilinear coordinates*

To compute the Green's function, we introduce a pair of convected oblique rectilinear contravariant coordinates, ξ and η, defined by

$$x = \xi + \tau \eta, \qquad y = \eta, \qquad (5.15.3)$$

where $\tau = at$. The inverse transformation is

$$\xi = x - \tau y, \qquad \eta = y. \qquad (5.15.4)$$

Since

$$\boldsymbol{\phi} \equiv \Big(\frac{\partial \mathbf{x}}{\partial t} \Big)_{\xi, \eta} = \mathbf{u}, \qquad (5.15.5)$$

the convected coordinates are confirmed to follow material point particles moving in straight paths under the influence of the simple shear flow. Comparing expressions (5.15.3) and (5.15.4) with those shown in (3.7.1) for canonical oblique rectilinear coordinates, we find that

$$\beta = \tau. \qquad (5.15.6)$$

We recall that $\beta(t) = \tan \phi$, where the evolving angle ϕ is defined in Figure 4.6.1.

The deformation gradient is given by

$$\mathbf{F} \equiv \left[\begin{array}{cc} \partial x / \partial \xi & \partial x / \partial \eta \\ \partial y / \partial \xi & \partial y / \partial \eta \end{array} \right] = \left[\begin{array}{cc} 1 & \tau \\ 0 & 1 \end{array} \right]. \qquad (5.15.7)$$

The inverse and the transpose of the inverse of the deformation gradient are given by

$$\mathbf{F}^{-1} = \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{F}^{-\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix}. \tag{5.15.8}$$

Setting $\mathbf{D} = \mathcal{D}\mathbf{I}$ for isotropic diffusion, we compute from (5.14.4)

$$\widehat{\mathbf{D}} = \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-\mathrm{T}} = \mathcal{D} \begin{bmatrix} 1+\tau^2 & -\tau \\ -\tau & 1 \end{bmatrix}, \tag{5.15.9}$$

where $\mathcal{D}$ is the scalar diffusivity.

The Green's function can be regarded either a function of x, y, t or a function of ξ, η, t,

$$\mathcal{G}(x, y, t) = \mathcal{Q}(\xi, \eta, t). \tag{5.15.10}$$

Using the chain rule, we find that

$$\frac{\partial \mathcal{G}}{\partial t} = \frac{\partial \mathcal{Q}}{\partial t} + \frac{\partial \mathcal{Q}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \mathcal{Q}}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial \mathcal{Q}}{\partial t} - a\eta\,\frac{\partial \mathcal{Q}}{\partial \xi}. \tag{5.15.11}$$

Similarly, we find that

$$\frac{\partial \mathcal{G}}{\partial x} = \frac{\partial \mathcal{Q}}{\partial \xi}. \tag{5.15.12}$$

Substituting these expressions into (5.15.2) and using expression (3.7.5) for the Laplacian in oblique rectilinear coordinates, repeated below for convenience,

$$\nabla^2 f = (1+\beta^2)\,\frac{\partial^2 f}{\partial \xi^2} - 2\,\beta\,\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}, \tag{5.15.13}$$

we derive a diffusion equation,

$$\frac{1}{\mathcal{D}}\,\frac{\partial \mathcal{Q}}{\partial t} = (1+\tau^2)\,\frac{\partial^2 \mathcal{Q}}{\partial \xi^2} - 2\,\tau\,\frac{\partial^2 \mathcal{Q}}{\partial \xi \partial \eta} + \frac{\partial^2 \mathcal{Q}}{\partial \eta^2} + \frac{1}{\mathcal{D}}\,\delta(\xi)\,\delta(\eta)\,\delta(t) \tag{5.15.14}$$

which can be expressed in the compact form

$$\frac{\partial \mathcal{Q}}{\partial t} = \tilde{\boldsymbol{\nabla}} \cdot (\widehat{\mathbf{D}} \cdot \widehat{\boldsymbol{\nabla}} \mathcal{Q}) + \delta(\xi)\, \delta(\eta)\, \delta(t), \qquad (5.15.15)$$

where $\widehat{\mathbf{D}}$ is given in (5.15.9) and $\widehat{\boldsymbol{\nabla}} = (\partial/\partial\xi, \partial/\partial\eta)$.

5.15.2 Solution in convected coordinates

To solve equation (5.15.15), we follow the procedure outlined in Section 5.6 and introduce the matrix

$$\mathbf{L}(t) \equiv 2 \int_0^t \widehat{\mathbf{D}}(t')\, \mathrm{d}t' = 2\, t\, \mathcal{D} \begin{bmatrix} 1 + \frac{1}{3}\tau^2 & -\frac{1}{2}\tau \\ -\frac{1}{2}\tau & 1 \end{bmatrix}, \qquad (5.15.16)$$

its determinant

$$\det(\mathbf{L}) = (2t\mathcal{D})^2\, (1 + \tfrac{1}{12}\tau^2), \qquad (5.15.17)$$

and its inverse,

$$\mathbf{L}^{-1} = \frac{1}{\det(\mathbf{L})} \begin{bmatrix} 1 & \frac{1}{2}\tau \\ \frac{1}{2}\tau & 1 + \frac{1}{3}\tau^2 \end{bmatrix}. \qquad (5.15.18)$$

According to (5.14.10), the solution of equation (5.15.14) is given by

$$\mathcal{Q}(\xi, \eta, t) = \frac{1}{2\pi \sqrt{\det(\mathbf{L})}}\, \exp(-\frac{1}{2\mathcal{D}}\, \boldsymbol{\xi} \cdot \mathbf{L}^{-1} \cdot \boldsymbol{\xi}), \qquad (5.15.19)$$

where $\boldsymbol{\xi} = (\xi, \eta)$. Making substitutions, we find that

$$\mathcal{Q}(\xi, \eta, t) = \frac{1}{4\pi\mathcal{D}t}\, \frac{1}{\sqrt{1 + \frac{1}{12}\tau^2}} \times \exp\big(-\frac{1}{4\mathcal{D}t}\, \frac{(\xi + \frac{1}{2}\tau\eta)^2}{1 + \frac{1}{12}\tau^2}\big) \exp\big(-\frac{1}{4\mathcal{D}t}\, \eta^2\big). \qquad (5.15.20)$$

Substituting the expressions for ξ and η in terms of x, y and t, we obtain the Green's function

$$\mathcal{G}(x, y, t) = \frac{1}{4\pi\mathcal{D}t}\, \frac{1}{\sqrt{1 + \frac{1}{12}a^2 t^2}} \times \exp\big(-\frac{1}{4\mathcal{D}t}\, \frac{(x - \frac{1}{2}aty)^2}{1 + \frac{1}{12}a^2t^2}\big) \exp\big(-\frac{1}{4\mathcal{D}t}\, y^2\big). \qquad (5.15.21)$$

In the absence of shear flow, $a = 0$, we obtain a well-known expression for the Green's function of the unsteady heat conduction equation.

Exercise

5.15.1 Derive (5.15.20) from (5.15.21).

5.16 Point source in oscillatory shear flow

As a further application, we consider the temperature field generated by a two-dimensional point source of heat located at the origin of the xy plane in the presence of an oscillatory simple shear flow. The velocity components along the x and y axes are given by

$$u_x = a \sin(\omega t + \phi)\, y, \qquad u_y = 0, \tag{5.16.1}$$

where a is the amplitude of the shear rate, ω is the angular frequency, and ϕ is the phase shift. The source is activated impulsively at the origin of time, $t = 0$, by sending an electrical signal through a wire.

The induced temperature field, $\mathcal{G}(x, y, t)$, is the Green's function of the associated singularly forced two-dimensional convection–diffusion equation,

$$\frac{\partial \mathcal{G}}{\partial t} + a\, \sin(\omega t + \phi)\, y\, \frac{\partial \mathcal{G}}{\partial x} = \mathcal{D}\,(\, \frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial y^2}\,) + \delta(x)\,\delta(y)\,\delta(t), \tag{5.16.2}$$

where δ is the one-dimensional Dirac delta function and $\mathcal{D}$ is the medium thermal diffusivity.

5.16.1 Convected coordinates

To find the solution, we introduce a pair of convected rectilinear coordinates, ξ and η, defined by

$$x = \xi + A \cos(\omega t + \phi)\, \eta, \qquad y = \eta, \tag{5.16.3}$$

where $A = a/\omega$ is a dimensionless parameter. The inverse transformation is

$$\xi = x - A\cos(\omega t + \phi)\, y, \qquad \eta = y. \tag{5.16.4}$$

The Green's function can be regarded as a function of x, y, t, or ξ, η, t,

$$\mathcal{G}(x, y, t) = \mathcal{Q}(\xi, \eta, t), \tag{5.16.5}$$

where ξ and η are functions of x, y, and t, by way of (5.16.4).

5.16.2 Solution in convected coordinates

Following the discussion in Section 5.7 of a point source in steady simple shear flow, we introduce the matrix

$$\widehat{\mathbf{D}} = \mathcal{D} \begin{bmatrix} 1 + A^2\cos^2(\tau + \phi) & -A\cos(\tau + \phi) \\ -A\cos(\tau + \phi) & 1 \end{bmatrix}, \tag{5.16.6}$$

where $\tau = \omega t$, and compute the matrix

$$\mathbf{L}(t) \equiv 2\int_0^t \widehat{\mathbf{D}}(t')\,\mathrm{d}t' = \frac{2}{\omega}\mathcal{D}\int_0^\tau \widehat{\mathbf{D}}(\tau')\,\mathrm{d}\tau'. \tag{5.16.7}$$

Performing the integration, we obtain

$$\mathbf{L}(t) = \frac{2}{\omega}\mathcal{D}\,\mathbf{\Lambda}, \tag{5.16.8}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \Lambda_{11} & -A\left(\sin(\tau + \phi) - \sin\phi\right) \\ -A\left(\sin(\tau + \phi) - \sin\phi\right) & \tau \end{bmatrix} \tag{5.16.9}$$

with

$$\Lambda_{11} = (1 + \tfrac{1}{2}A^2)\,\tau + \tfrac{1}{4}A^2\left(\sin(2\tau + 2\phi) - \sin(2\phi)\right). \tag{5.16.10}$$

The solution is given by formula (5.15.19), repeated below for convenience,

$$\mathcal{Q}(\xi, \eta, t) = \frac{1}{2\pi\sqrt{\det(\mathbf{L})}}\,\exp(-\frac{1}{2\mathcal{D}}\,\boldsymbol{\xi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\xi}), \tag{5.16.11}$$

where $\boldsymbol{\xi} = (\xi, \eta)$. The Green's function arises by using equations (5.16.4) to express ξ and η in terms of x and y.

Exercise

5.16.1 Derive the expression for $\mathbf{\Lambda}$ shown in (5.16.9).

5.17 Point source in extensional flow

As a last application of convected coordinates, we consider the temperature field generated by a two-dimensional point source of heat located at the origin of the xy plane, in the presence of a steady extensional flow. The velocity components along the x and y axes are given by

$$u_x = ax, \qquad u_y = -ay, \tag{5.17.1}$$

where a is a constant identified as the extensional rate. The source is activated impulsively at the origin of time, $t = 0$.

The induced temperature field, $\mathcal{G}$, identified as a Green's function, satisfies the two-dimensional singularly forced convection–diffusion equation

$$\frac{\partial \mathcal{G}}{\partial t} + ax\,\frac{\partial \mathcal{G}}{\partial x} - ay\,\frac{\partial \mathcal{G}}{\partial y} = \mathcal{D}\,(\,\frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial y^2}\,) + \delta(x)\,\delta(y)\,\delta(t), \tag{5.17.2}$$

where δ is the one-dimensional Dirac delta function and $\mathcal{D}$ is the medium thermal diffusivity.

5.17.1 Convected coordinates

To find the solution, we introduce a pair of convected rectilinear coordinates, ξ and η, defined by

$$x = \xi\,\mathrm{e}^{at} \qquad y = \eta\,\mathrm{e}^{-at}. \tag{5.17.3}$$

The inverse relations are

$$\xi = x\,\mathrm{e}^{-at} \qquad \eta = y\,\mathrm{e}^{at}. \tag{5.17.4}$$

Since

$$\boldsymbol{\phi} = \Big(\frac{\partial \mathbf{x}}{\partial t}\Big)_{\xi,\eta} = \mathbf{u}, \tag{5.17.5}$$

the convected coordinates are the trajectories of material point particles moving under the influence of the extensional flow. The Green's function can be regarded as a function of ξ, η, and t,

$$\mathcal{G}(x,y,t) = \mathcal{Q}(\xi,\eta,t), \tag{5.17.6}$$

where ξ and η are functions of x, y, and t by way of (5.17.4).

5.17.2 Solution in convected coordinates

Substituting the preceding expressions into the governing equation (5.17.2), we derive an anisotropic diffusion equation for Q,

$$\frac{1}{\mathcal{D}}\,\frac{\partial \mathcal{Q}}{\partial t} = \mathrm{e}^{-2at}\,\frac{\partial^2 \mathcal{Q}}{\partial \xi^2} + \mathrm{e}^{2at}\,\frac{\partial^2 \mathcal{Q}}{\partial \eta^2} + \frac{1}{\mathcal{D}}\,\delta(\xi)\,\delta(\eta)\,\delta(t). \tag{5.17.7}$$

The solution is given by formula (5.15.19), repeated below for convenience,

$$\mathcal{Q}(\xi,\eta,t) = \frac{1}{2\pi\sqrt{\det(\mathbf{L})}}\,\exp(-\frac{1}{2\mathcal{D}}\,\boldsymbol{\xi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\xi}), \tag{5.17.8}$$

where $\boldsymbol{\xi} = (\xi,\eta)$,

$$\widehat{\mathbf{D}} = \mathcal{D}\left[\begin{array}{cc} \mathrm{e}^{-2at} & 0 \\ 0 & \mathrm{e}^{2at} \end{array}\right] \tag{5.17.9}$$

and

$$\mathbf{L} \equiv 2\int_0^t \widehat{\mathbf{D}}(t')\,\mathrm{d}t' = \frac{1}{a}\,\mathcal{D}\left[\begin{array}{cc} 1-\mathrm{e}^{-2at} & 0 \\ 0 & \mathrm{e}^{2at}-1 \end{array}\right]. \tag{5.17.10}$$

The Green's function arises by using equations (5.17.4) to express ξ and η in terms of x and y.

Exercise

5.17.1 Compute and discuss the time dependence of the determinant of the matrix $\mathbf{L}$.

Chapter 6

Surface coordinates

A pair of orthogonal or non-orthogonal, rectilinear or curvilinear surface coordinates and associated covariant base vectors can be defined over a flat or curved surface embedded in three-dimensional space. In science and engineering applications, such surfaces are typically identified with fluid or solid boundaries or interfaces, biological or manufactured membranes, and thin shells. The mathematical apparatus of curvilinear coordinates discussed in Chapters 4 and 5 in a plane or three-dimensional space can be adapted to describe surfaces in terms of surface coordinates.

The analysis leads us naturally to the notion of the surface curvature tensor and to the concept of the Christoffel–Riemann curvature tensor. Expressions for the directional surface derivative of a scalar, the surface divergence and gradient of a vector or tensor field, and other surface differential operators can be derived following familiar steps. The derivations of such expressions will be pursued and applications will be discussed in this chapter with reference to the equilibrium shapes of (a) membranes developing tangential tensions and (b) thin shells with small but infinitesimal thickness developing tangential tensions, transverse shear tensions, and accompanying bending moments.

6.1 Parametric description and base vectors

A pair of surface curvilinear coordinates, (x^1, x^2), can be introduced to describe parametrically a surface in terms of the position vector in three-dimensional space, as

$$\mathbf{x}(x^1, x^2), \tag{6.1.1}$$

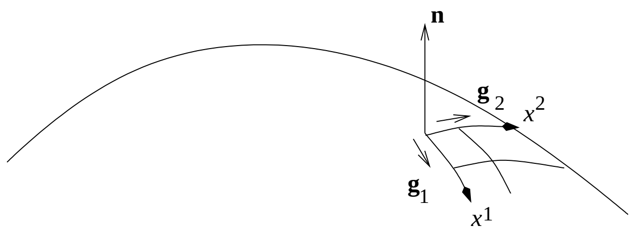

FIGURE 6.1.1 Illustration of curvilinear coordinates in a surface embedded in three-dimensional space, showing the surface covariant base vectors, $(\mathbf{g}_1, \mathbf{g}_2)$, associated surface contravariant coordinates, (x^1, x^2), and the unit normal vector, $\mathbf{n}$.

as shown in Figure 6.1.1. For example, the equations

$$x = a\cos\xi, \qquad y = b\sin\xi\cos\eta, \qquad z = c\sin\xi\sin\eta, \quad (6.1.2)$$

describe the surface of a spheroid, where a, b, c are the spheroid semi-axes and ξ and η are two surface parameters that can be regarded as surface curvilinear coordinates by setting $\xi = x^1$ and $\eta = x^2$.

6.1.1 Surface base vectors

The covariant surface base vectors,

$$\mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial x^1}, \qquad \mathbf{g}_2 = \frac{\partial \mathbf{x}}{\partial x^2}, \qquad (6.1.3)$$

are tangential to the surface and thus perpendicular to the unit normal vector, $\mathbf{n}$, at any point,

$$\mathbf{g}_1 \cdot \mathbf{n} = 0, \qquad \mathbf{g}_2 \cdot \mathbf{n} = 0. \qquad (6.1.4)$$

The unit normal vector can be expressed in terms of the outer product of the two surface base vectors,

$$\mathbf{n} = \frac{1}{|\mathbf{g}_1 \times \mathbf{g}_2|}\,\mathbf{g}_1 \times \mathbf{g}_2. \qquad (6.1.5)$$

The covariant surface metric coefficients,

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j, \qquad (6.1.6)$$

can be accommodated in the 2×2 matrix denoted by $\mathbf{g}$.

6.1.2 Surface metric

The surface metric coefficient associated with the covariant base vectors is given by

$$\mathcal{J} = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{n} = |\mathbf{g}_1 \times \mathbf{g}_2|. \tag{6.1.7}$$

The area of an infinitesimal surface parallelepiped whose sides are described by the differential displacements $\mathrm{d}x^1$ and $\mathrm{d}x^2$ is given by

$$\mathrm{d}S = \mathcal{J}\, \mathrm{d}x^1 \mathrm{d}x^2. \tag{6.1.8}$$

Recalling that the determinant of a matrix is equal to the determinant of the matrix transpose, we write

$$\mathcal{J}^2 = \det\Bigg(\begin{bmatrix} (\mathbf{g}_1)_x & (\mathbf{g}_1)_y & (\mathbf{g}_1)_z \\ (\mathbf{g}_2)_x & (\mathbf{g}_2)_y & (\mathbf{g}_2)_z \\ n_x & n_y & n_z \end{bmatrix} \cdot \begin{bmatrix} (\mathbf{g}_1)_x & (\mathbf{g}_2)_x & n_x \\ (\mathbf{g}_1)_y & (\mathbf{g}_2)_y & n_y \\ (\mathbf{g}_1)_z & (\mathbf{g}_2)_z & n_z \end{bmatrix} \Bigg), \tag{6.1.9}$$

and obtain

$$\mathcal{J}^2 = \det\Bigg(\begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg). \tag{6.1.10}$$

We thus find that

$$\mathcal{J}^2 = g, \tag{6.1.11}$$

where

$$g \equiv \det(\mathbf{g}) = g_{11} g_{22} - g_{12}^2 \tag{6.1.12}$$

is the determinant of the matrix of covariant metric surface coefficient. Combining (6.1.7) with (6.1.11), we find that

$$\sqrt{g} = |\mathbf{g}_1 \times \mathbf{g}_2|, \tag{6.1.13}$$

which requires that g be positive.

6.1.3 Contravariant surface base vectors

The contravariant surface base vectors, $\mathbf{g}^1$ and $\mathbf{g}^2$, are given by

$$\mathbf{g}^1 = \frac{1}{\mathcal{J}}\,\mathbf{g}_2 \times \mathbf{n}, \qquad \mathbf{g}^2 = \frac{1}{\mathcal{J}}\,\mathbf{n} \times \mathbf{g}_1. \tag{6.1.14}$$

Substituting $\mathbf{n} = (1/\mathcal{J})\,\mathbf{g}_1 \times \mathbf{g}_2$ and using the properties of the double outer product, we obtain

$$\mathbf{g}^1 = \frac{1}{\mathcal{J}^2}\big(\,|\mathbf{g}_2|^2\,\mathbf{g}_1 - (\mathbf{g}_1 \cdot \mathbf{g}_2)\,\mathbf{g}_2)\,\big) \tag{6.1.15}$$

and

$$\mathbf{g}^2 = \frac{1}{\mathcal{J}^2}\big(\,|\mathbf{g}_1|^2\,\mathbf{g}_2 - (\mathbf{g}_1 \cdot \mathbf{g}_2)\,\mathbf{g}_1)\,\big). \tag{6.1.16}$$

Like the covariant surface vectors, the contravariant surface vectors are also tangential to the surface and thus perpendicular to the unit normal vector,

$$\mathbf{g}^1 \cdot \mathbf{n} = 0, \qquad \mathbf{g}^2 \cdot \mathbf{n} = 0. \tag{6.1.17}$$

Conversely, the covariant surface vectors can be recovered from the contravariant surface vectors using the relations

$$\mathbf{g}_1 = \mathcal{J}\,\mathbf{g}^2 \times \mathbf{n}, \qquad \mathbf{g}_2 = \mathcal{J}\,\mathbf{n} \times \mathbf{g}^1. \tag{6.1.18}$$

By construction, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_{ij}$ for $i, j = 1, 2$, where δ_{ij} is Kronecker's delta.

6.1.4 Contravariant metric coefficients

The surface contravariant metric coefficients,

$$g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j \tag{6.1.19}$$

for $i, j = 1, 2$, can be accommodated in the 2×2 matrix denoted by $\boldsymbol{\gamma}$. Working as in Chapter 4, we find that the matrix of covariant surface coefficients is the inverse of the matrix of the contravariant surface coefficients, and *vice versa*

$$\boldsymbol{\gamma} = \mathbf{g}^{-1}. \tag{6.1.20}$$

Consequently,

$$g^{11} = \frac{g_{22}}{g}, \qquad g^{22} = \frac{g_{11}}{g}, \qquad g^{12} = -\frac{g_{12}}{g}, \tag{6.1.21}$$

where $g = \det(\mathbf{g})$.

6.1.5 One-third orthogonality

The triplet $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{n})$ defines a system of partially orthogonal local directions at a point on a surface, as shown in Figure 6.1.1. The triplet $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{n})$ defines another system of partially orthogonal local directions. We may introduce the arc length measured normal to the surface at a point, x^3, and regard x^1, x^2, x^3 and x_1, x_2, x^3 as partially orthogonal spatial coordinates defined over the surface and extended off the surface into three-dimensional space.

6.1.6 First fundamental form of a surface

The differential of the position vector in a surface can be expressed in terms of the surface covariant base vectors as

$$\mathrm{d}\mathbf{x} = \mathbf{g}_i \, \mathrm{d}x^i, \tag{6.1.22}$$

where summation is implied over the repeated index, i. The first fundamental form of the surface is an expression for the square of the length of the differential vector,

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = g_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j, \tag{6.1.23}$$

where summation is implied over the repeated indices, i and j.

In standard differential geometry notation,

$$E \equiv g_{11}, \qquad F \equiv g_{12} = g_{21}, \qquad G \equiv g_{22}. \tag{6.1.24}$$

Consequently

$$g \equiv \det(\mathbf{g}) = EG - F^2. \tag{6.1.25}$$

Exercises

6.1.1 Derive expressions for the covariant base vectors over a spheroid.

6.1.2 Prove relation (6.1.20).

6.2 Projection tensor

To extract the tangential component of a general vector or operator defined over a surface, we multiply it with a projection tensor defined as

$$\mathbf{P} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}, \tag{6.2.1}$$

where $\mathbf{I}$ is the identity matrix. In index notation,

$$P_{ij} = \delta_{ij} - n_i n_j. \tag{6.2.2}$$

The projection tensor is sometimes called the *surface identity tensor.* By construction, the surface projection tensor is symmetric, that is, it is equal to its transpose.

Since $\mathbf{P} \cdot \mathbf{n} = \mathbf{0}$, the normal vector, $\mathbf{n}$, is an eigenvector of $\mathbf{P}$ with zero corresponding eigenvalue. Any tangential vector is an eigenvector with unity corresponding eigenvalue.

6.2.1 Conjugate normal planes

We may introduce two perpendicular normal planes at a chosen point defined by two mutually orthogonal unit tangential vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, as shown in Figure 6.2.1, where $|\mathbf{t}_1| = 1$, $|\mathbf{t}_2| = 1$, and

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = 0. \tag{6.2.3}$$

The projection tensor is given by

$$\mathbf{P} = \mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2 \tag{6.2.4}$$

for any pair $\mathbf{t}_1$ and $\mathbf{t}_2$ subject to the aforementioned restrictions. We may confirm that the normal vector, $\mathbf{n}$, is an eigenvector of $\mathbf{P}$ with zero corresponding eigenvalue, $\mathbf{P} \cdot \mathbf{n} = \mathbf{0}$. Moreover,

$$\mathbf{P} \cdot \mathbf{t}_1 = \mathbf{t}_1, \qquad \mathbf{P} \cdot \mathbf{t}_2 = \mathbf{t}_2, \tag{6.2.5}$$

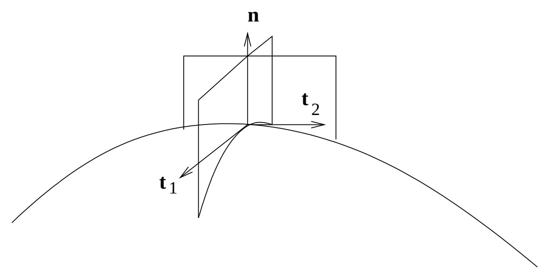

FIGURE 6.2.1 Illustration of two perpendicular normal planes defined by two mutually orthogonal unit tangential vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$.

which shows that $\mathbf{t}_1$ and $\mathbf{t}_2$ are eigenvectors with corresponding unit eigenvalues.

6.2.2 Normal and tangential vector components

The normal component of an arbitrary vector, $\mathbf{a}$, defined over a surface is $\mathbf{a} \cdot \mathbf{n}$, and the tangential component is

$$\mathbf{P} \cdot \mathbf{a} = \mathbf{n} \times \mathbf{a} \times \mathbf{n}. \tag{6.2.6}$$

The equivalence of the two expressions in this equation can be proved readily working in index notation. Consequently, we may write

$$\mathbf{a} = \mathbf{n}\,(\mathbf{n} \cdot \mathbf{a}) + \mathbf{P} \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{n} \otimes \mathbf{n}) + \mathbf{P} \cdot \mathbf{a}, \tag{6.2.7}$$

expressing a normal-tangent decomposition.

6.2.3 Tangential gradient operator

The gradient operator, $\boldsymbol{\nabla}$, may likewise be resolved into normal and tangential components,

$$\boldsymbol{\nabla} = \mathbf{n}\,(\mathbf{n} \cdot \boldsymbol{\nabla}) + \mathbf{P} \cdot \boldsymbol{\nabla}\,, \tag{6.2.8}$$

where the normal component expresses a derivative along $\mathbf{n}$ and the tangential component encapsulates derivatives normal to $\mathbf{n}$. For convenience, we define the tangential component of the gradient

$$\widehat{\boldsymbol{\nabla}} \equiv \mathbf{P} \cdot \boldsymbol{\nabla}\,. \tag{6.2.9}$$

We will see that the tangential gradient can operate on scalars, vectors, and tensors as an inner product, outer product, or tensor product.

6.2.4 Projection tensor in surface coordinates

We may refer to surface curvilinear coordinates and corresponding base vectors, as shown in Figure 6.1.1, and expand the projection tensor as

$$\mathbf{P} = P^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j, \tag{6.2.10}$$

where $\mathbf{g}_j$ are covariant surface base vectors and P^{ij} are the contravariant components of $\mathbf{P}$.

Projecting equation (6.2.10) onto a covariant base vector, $\mathbf{g}^m$, where m is a free index, and recalling that $\mathbf{g}_j \cdot \mathbf{g}^m = \delta_{jm}$ and $\mathbf{P} \cdot \mathbf{g}^m = \mathbf{g}^m$, we find that

$$\mathbf{g}^m = P^{im}\,\mathbf{g}_i. \tag{6.2.11}$$

Projecting this equation onto $\mathbf{g}^n$, where n is another free index, we find that

$$\mathbf{g}^m \cdot \mathbf{g}^n \equiv g^{mn} = P^{nm}. \tag{6.2.12}$$

This expression shows that

$$\mathbf{P} = g^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j. \tag{6.2.13}$$

In the case of orthogonal coordinates, we recover (6.2.4).

Working in a similar fashion, we find that the tangential projection tensor can be expanded in four ways as

$$\mathbf{P} = g^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j = g_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i, \tag{6.2.14}$$

where summation is implied over the repeated indices, i and j. These expansions reveal that the projection tensor $\mathbf{P}$ is, in fact, the surface metric tensor, which is the counterpart of the identity tensor in two or three dimensions.

Exercise

6.2.1 Derive the last two expressions in (6.2.14).

6.3 Surface curvatures

Consider the planar intersection of a surface with each one of the two mutual perpendicular planes defined by the unit vectors $\mathbf{t}_1$ or $\mathbf{t}_2$, as shown in Figure 6.2.1, where $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$. Now introduce the arc lengths, ℓ_1 and ℓ_2, measured over the surface along the intersections in the directions of $\mathbf{t}_1$ or $\mathbf{t}_2$.

6.3.1 Local variation of the normal vector

The derivatives of the unit normal vector with respect to arc length, $\partial\mathbf{n}/\partial\ell_1$ and $\partial\mathbf{n}/\partial\ell_2$, are tangential vectors with components along $\mathbf{t}_1$ and $\mathbf{t}_2$, but not along $\mathbf{n}$,

$$\frac{\partial \mathbf{n}}{\partial \ell_1} = \kappa_1\,\mathbf{t}_1 + \kappa_{12}\,\mathbf{t}_2, \qquad \frac{\partial \mathbf{n}}{\partial \ell_2} = \kappa_{21}\,\mathbf{t}_1 + \kappa_2\,\mathbf{t}_2, \tag{6.3.1}$$

where κ_1, κ_2, κ_{12}, and κ_{21} are curvature coefficients. To explain the absence of $\mathbf{n}$ from the right-hand side of (6.3.1), we note that, because the normal vector $\mathbf{n}$ is a unit vector,

$$\frac{\partial(\mathbf{n}\cdot\mathbf{n})}{\partial \ell_1} = 0, \qquad \frac{\partial(\mathbf{n}\cdot\mathbf{n})}{\partial \ell_2} = 0, \tag{6.3.2}$$

and thus

$$\mathbf{n}\cdot\frac{\partial \mathbf{n}}{\partial \ell_1} = 0, \qquad \mathbf{n}\cdot\frac{\partial \mathbf{n}}{\partial \ell_2} = 0, \tag{6.3.3}$$

which shows that the normal components of the derivatives $\partial\mathbf{n}/\partial\ell_1$ and $\partial\mathbf{n}/\partial\ell_2$ are zero.

6.3.2 Curvature coefficients

Projecting the first equation in (6.3.1) on $\mathbf{t}_2$ and the second equation on $\mathbf{t}_1$, we find that

$$\kappa_{12} = \frac{\partial\,\mathbf{n}}{\partial \ell_1}\cdot\mathbf{t}_2, \qquad \kappa_{21} = \frac{\partial\,\mathbf{n}}{\partial \ell_2}\cdot\mathbf{t}_1. \tag{6.3.4}$$

Since $\mathbf{t}_1$ and $\mathbf{t}_2$ are fixed vectors regarded as unit vectors of a local Cartesian system, we may write

$$\kappa_{12} = \frac{\partial\,(\mathbf{n}\cdot\mathbf{t}_2)}{\partial \ell_1} = \frac{\partial\, n_2}{\partial \ell_1} \qquad \kappa_{21} = \frac{\partial\,(\mathbf{n}\cdot\mathbf{t}_1)}{\partial \ell_2} = \frac{\partial\, n_1}{\partial \ell_2}, \tag{6.3.5}$$

where n_1 is component of $\mathbf{n}$ in the direction of $\mathbf{t}_1$ and n_2 is component of $\mathbf{n}$ in the direction of $\mathbf{t}_2$.

Moreover, we may write $\mathbf{t}_2 = \partial\mathbf{x}/\partial\ell_2$ evaluated at the chose surface point, and then

$$\kappa_{12} = \frac{\partial\,\mathbf{n}}{\partial \ell_1}\cdot\frac{\partial\,\mathbf{x}}{\partial \ell_2} = \frac{\partial}{\partial \ell_1}\Big(\mathbf{n}\cdot\frac{\partial \mathbf{x}}{\partial \ell_2}\Big) - \mathbf{n}\cdot\frac{\partial^2\,\mathbf{x}}{\partial \ell_1 \partial \ell_2}. \tag{6.3.6}$$

Because $\mathbf{n}$ is normal to the tangential vector $\partial\mathbf{x}/\partial\ell_2$, the first term on the right-hand side is zero. Working similarly for κ_{21}, we derive an analogous expression and conclude that

$$\kappa_{12} = \kappa_{21} = -\mathbf{n}\cdot\frac{\partial^2\,\mathbf{x}}{\partial \ell_1 \partial \ell_2}. \tag{6.3.7}$$

6.3.3 Matrix of curvature coefficients

The four coefficients κ_1, κ_2, κ_{12}, and κ_{21} can be collected in a symmetric matrix denoted by

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 & \kappa_{12} \\ \kappa_{21} & \kappa_2 \end{bmatrix}. \tag{6.3.8}$$

Because $\boldsymbol{\kappa}$ is symmetric, it has real eigenvalues and an orthogonal pair of eigenvectors indicating the directions of principal curvatures in a tangential plane.

We will see that the trace and the determinant of this matrix express intrinsic properties of the surface in that they are independent of the orientation of the two perpendicular planes.

6.3.4 Directional curvatures

The coefficients κ_1 and κ_2 in (6.3.1) are the curvatures of the intersections of the surface with each one of the two mutual perpendicular planes. Projecting the first equation in (6.3.1) into $\mathbf{t}_1$ and the second equation onto $\mathbf{t}_2$, we find that

$$\kappa_1 = \frac{\partial\,\mathbf{n}}{\partial \ell_1} \cdot \mathbf{t}_1, \qquad \kappa_2 = \frac{\partial\,\mathbf{n}}{\partial \ell_2} \cdot \mathbf{t}_2. \tag{6.3.9}$$

Since $\mathbf{t}_1$ and $\mathbf{t}_2$ are fixed vectors, we may write

$$\kappa_1 = \frac{\partial\,(\mathbf{n}\cdot\mathbf{t}_1)}{\partial \ell_1} = \frac{\partial\,n_1}{\partial \ell_1} \qquad \kappa_2 = \frac{\partial\,(\mathbf{n}\cdot\mathbf{t}_2)}{\partial \ell_2} = \frac{\partial\,n_2}{\partial \ell_2}, \tag{6.3.10}$$

where n_1 is component of $\mathbf{n}$ in the direction of $\mathbf{t}_1$ and n_2 is component of $\mathbf{n}$ in the direction of $\mathbf{t}_2$.

6.3.5 Arbitrary directional curvature

It is helpful to introduce the tangential differentiation operators

$$\mathbf{t}_1 \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial \ell_1}, \qquad \mathbf{t}_2 \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial \ell_2}, \tag{6.3.11}$$

and write

$$\kappa_1 = (\mathbf{t}_1 \cdot \boldsymbol{\nabla}\,\mathbf{n}) \cdot \mathbf{t}_1, \qquad \kappa_2 = (\mathbf{t}_2 \cdot \boldsymbol{\nabla}\,\mathbf{n}) \cdot \mathbf{t}_2, \tag{6.3.12}$$

and

$$\kappa_{12} = (\mathbf{t}_1 \cdot \boldsymbol{\nabla}\,\mathbf{n}) \cdot \mathbf{t}_2, \qquad \kappa_{21} = (\mathbf{t}_2 \cdot \boldsymbol{\nabla}\,\mathbf{n}) \cdot \mathbf{t}_1, \tag{6.3.13}$$

where ℓ_1 and ℓ_2 are arc lengths measured along the intersections in the directions of $\mathbf{t}_1$ or $\mathbf{t}_2$.

Now we consider an arbitrary unit tangent vector, $\mathbf{t}_\lambda$, in the plane of $\mathbf{t}_1$ and $\mathbf{t}_2$, given by

$$\mathbf{t}_\lambda = \frac{1}{\sqrt{1+\lambda^2}}\,(\mathbf{t}_1 + \lambda \mathbf{t}_2), \tag{6.3.14}$$

where λ is a an arbitrary positive parameter. The orthogonality constraint $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$ guarantees that $\mathbf{t}_\lambda$ is a unit vector, $\mathbf{t}_\lambda \cdot \mathbf{t}_\lambda = 1$. The curvature of the intersection of the surface with the corresponding normal vector is given by

$$\kappa_\lambda = \frac{\partial\,\mathbf{n}}{\partial \ell_\lambda} \cdot \mathbf{t}_\lambda = (\mathbf{t}_\lambda \cdot \boldsymbol{\nabla}\,\mathbf{n}) \cdot \mathbf{t}_\lambda, \tag{6.3.15}$$

where ℓ_λ is the arc length measured in the direction of $\mathbf{t}_\lambda$. Substituting (6.3.14) into the right-hand side, and using (6.3.12) and (6.3.13), we obtain

$$\kappa_\lambda = \frac{1}{1+\lambda^2}\,(\kappa_1 + \lambda^2\kappa_2 + \lambda\,2\kappa_{12}). \tag{6.3.16}$$

When $\lambda = 0$ or ∞, we recover κ_1 and κ_2.

6.3.6 Mean curvature

The sum of curvatures in two directions corresponding to two arbitrary values, λ_1 and λ_2, is

$$\kappa_{\lambda_1} + \kappa_{\lambda_2} = \frac{1}{1+\lambda_1^2}\,(\kappa_1 + \lambda_1^2\kappa_2 + \lambda_1\,2\kappa_{12}) + \frac{1}{1+\lambda_2^2}\,(\kappa_1 + \lambda_2^2\kappa_2 + \lambda_2\,2\kappa_{12}). \tag{6.3.17}$$

Referring to (6.3.14), we find that the directions corresponding to two values, λ_1 and λ_2, are mutually orthogonal if $\lambda_1\lambda_2 = -1$. Substituting into (6.3.17) $\lambda_2 = -1/\lambda_1$, we obtain

$$\kappa_{\lambda_1} + \kappa_{\lambda_2} = \frac{1}{1+\lambda_1^2}\,(\kappa_1 + \lambda_1^2\kappa_2 + \lambda_1\,2\kappa_{12}) + \frac{\lambda_1^2}{1+\lambda_1^2}\,(\kappa_1 + \frac{1}{\lambda_1^2}\,\kappa_2 - \frac{1}{\lambda_1}\,2\kappa_{12}). \tag{6.3.18}$$

Performing the computations on the right-hand side, we find that

$$\kappa_{\lambda_1} + \kappa_{\lambda_2} = \kappa_1 + \kappa_2, \tag{6.3.19}$$

independent of λ_1 or λ_2, provided that $\lambda_1\lambda_2 = -1$.

We have found that the mean curvature of the surface at an arbitrary point is

$$\kappa_{\text{mean}} = \tfrac{1}{2}\,\text{trace}(\boldsymbol{\kappa}) = \tfrac{1}{2}\,(\kappa_1 + \kappa_2), \tag{6.3.20}$$

independent of the orientation of the two normal planes.

6.3.7 Principal curvatures

To identify the maximum and minimum curvatures, comprising the principal curvatures, we set $\mathrm{d}\kappa_\lambda/\mathrm{d}\lambda = 0$ and differentiate (6.3.16) with respect to λ to obtain the equation

$$-2\lambda\,(\kappa_1 + \lambda^2 \kappa_2 + \lambda\,2\kappa_{12}) + (1+\lambda^2)\,(2\lambda\kappa_2 + 2\kappa_{12}) = 0. \quad (6.3.21)$$

Simplifying, we obtain a quadratic equation for λ,

$$\lambda^2 + \lambda\,\frac{\kappa_1 - \kappa_2}{\kappa_{12}} - 1 = 0. \quad (6.3.22)$$

The roots can be computed readily using the quadratic formula. The product of the roots is $\lambda_1\lambda_2 = -1$, corresponding to mutually orthogonal orientations. The sum of the roots is

$$\lambda_1 + \lambda_2 = \frac{\kappa_2 - \kappa_1}{\kappa_{12}}. \quad (6.3.23)$$

When $\kappa_{12} = 0$, the roots are $\lambda = 0, \infty$. Conversely, if κ_1 and κ_2 are the principal curvatures, then $\kappa_{12} = 0$.

6.3.8 Confirmation by code

The following Matlab code named *curvatures*, located in directory CURVATURES of TUNLIB, confirms that the principal curvatures computed in this fashion are the eigenvalues of the curvature tensor $\boldsymbol{\kappa}$:

```
kappa1  =  1.4; % arbitrary
kappa2  = -0.2; % arbitrary
kappa12 =  3.0; % arbitrary

kappa = [kappa1,  kappa12;
         kappa12, kappa2];

kappap = eig(kappa);

b = (kappa1-kappa2)/kappa12;
disc = sqrt(b^2+4);
lam1 = (-b-disc)/2;
lam2 = (-b+disc)/2;
```

```
kappap1 = (kappa1+lam1^2*kappa2+lam1*2*kappa12)/(1+lam1^2);
kappap2 = (kappa1+lam2^2*kappa2+lam2*2*kappa12)/(1+lam2^2);

[kappap(1), kappap(2); kappap1, kappap2]
```

Running the code generates the following output:

```
   -2.5048    3.7048
   -2.5048    3.7048
```

as prompted by the last line of the code. We see that the principal curvatures computed in two different ways are the same.

6.3.9 Mean and Gaussian curvatures

Let $\kappa_{\max}$ be the maximum curvature and $\kappa_{\min}$ be the minimum curvature. The mean curvature is

$$\kappa_{\text{mean}} = \tfrac{1}{2}\,(\kappa_{\max} + \kappa_{\min}). \tag{6.3.24}$$

The Gaussian curvature is defined as

$$H \equiv \kappa_{\max}\,\kappa_{\min}. \tag{6.3.25}$$

Using the formulas for the directional curvatures, we find that

$$H = \frac{1}{(1+\lambda_1^2)^2}\,(\kappa_1 + \lambda_1^2\kappa_2 + \lambda_1\,2\kappa_{12})\,(\lambda_1^2\kappa_1 + \kappa_2 - \lambda_1\,2\kappa_{12}), \tag{6.3.26}$$

where λ_1 is a root of (6.3.22). Simplifying, we obtain

$$H = \det(\boldsymbol{\kappa}) = \kappa_1\kappa_2 - \kappa_{12}^2. \tag{6.3.27}$$

We recall that, if κ_1 and κ_2 are the principal curvatures, then $\kappa_{12} = 0$.

Exercise

6.3.1 Explain why the Gaussian curvature of a sphere of radius a is $H = 1/a^2$, whereas the Gaussian curvature of a cylinder is zero. Discuss whether the Gaussian curvature of a torus is also zero.

6.4 Curvature tensor

The curvature tensor is defined with reference to the principal curvatures, $\kappa_{\max}$ and $\kappa_{\min}$, and associated unit tangential vectors $\mathbf{t}_{\max}$ and $\mathbf{t}_{\min}$, as

$$\mathbf{B} \equiv \kappa_{\max}\,\mathbf{t}_{\max} \otimes \mathbf{t}_{\max} + \kappa_{\min}\,\mathbf{t}_{\min} \otimes \mathbf{t}_{\min}. \tag{6.4.1}$$

By definition, the curvature tensor is symmetric, that is

$$\mathbf{B} = \mathbf{B}^{\mathrm{T}}, \tag{6.4.2}$$

where the superscript T denotes the matrix transpose.

In the literature of differential geometry, the negative of the curvature tensor is employed,

$$\mathbf{b} = -\mathbf{B}. \tag{6.4.3}$$

According to this alternative definition, the curvature of a sphere is negative, or else the direction of the unit normal vector, $\mathbf{n}$, is reversed to point toward the center.

By construction, the tangential vectors $\mathbf{t}_{\max}$ and $\mathbf{t}_{\min}$, and the unit normal vector, $\mathbf{n}$, are the eigenvectors of the curvature tensor with corresponding eigenvalues $\kappa_{\max}$, $\kappa_{\max}$, and 0,

$$\mathbf{B} \cdot \mathbf{t}_{\max} = \kappa_{\max}\,\mathbf{t}_{\max}, \qquad \mathbf{B} \cdot \mathbf{t}_{\min} = \kappa_{\min}\,\mathbf{t}_{\min},$$
$$\mathbf{B} \cdot \mathbf{n} = \mathbf{0}. \tag{6.4.4}$$

Because of the presence of a zero eigenvalue, the determinant of $\mathbf{B}$ is zero, that is, the curvature tensor is singular.

6.4.1 Mean curvature

The mean curvature of the surface at a point is half the trace of the curvature tensor at that point,

$$\kappa_m = \tfrac{1}{2}\,\mathrm{trace}(\mathbf{B}). \tag{6.4.5}$$

With reference to (6.4.1), because $\mathbf{t}_{\max}$ and $\mathbf{t}_{\min}$ are unit vectors, the traces of the tensors $\mathbf{t}_{\max} \otimes \mathbf{t}_{\max}$ and $\mathbf{t}_{\min} \otimes \mathbf{t}_{\min}$ are both equal to unity, while the trace of tensor $\mathbf{t}_{\max} \otimes \mathbf{t}_{\min}$, which is equal to the inner product $\mathbf{t}_{\max} \cdot \mathbf{t}_{\min}$, is zero.

6.4.2 Spectral decomposition

We may arrange the eigenvectors at the columns of an orthogonal matrix

$$\mathbf{U} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{t}_{\max} & \mathbf{t}_{\min} & \mathbf{n} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \tag{6.4.6}$$

and construct the curvature tensor as

$$\mathbf{B} = \mathbf{U} \cdot \mathbf{\Lambda} \cdot \mathbf{U}^{\mathrm{T}}, \tag{6.4.7}$$

where

$$\mathbf{\Lambda} \equiv \begin{bmatrix} \kappa_{\max} & 0 & 0 \\ 0 & \kappa_{\min} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.4.8}$$

is the diagonal matrix of eigenvalues. Orthogonality ensures that $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$.

6.4.3 Curvature tensor in arbitrary orthogonal tangential coordinates

With reference to Figure 6.2.1 and an arbitrary pair of mutually orthogonal unit vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, the curvature tensor takes the extended form

$$\mathbf{B} \equiv \kappa_1 \, \mathbf{t}_1 \otimes \mathbf{t}_1 + \kappa_2 \, \mathbf{t}_2 \otimes \mathbf{t}_2 + \kappa_{12} \, \mathbf{t}_1 \otimes \mathbf{t}_2 + \kappa_{21} \, \mathbf{t}_2 \otimes \mathbf{t}_1. \tag{6.4.9}$$

Because $\kappa_{12} = \kappa_{21}$, as discussed in Section 6.3, the symmetry of the curvature tensor is guaranteed.

6.4.4 Curvature tensor in terms of the surface gradient of the normal vector

Using equations (6.3.1), we find that

$$\mathbf{t}_1 \cdot \mathbf{B} = \kappa_1 \, \mathbf{t}_1 + \kappa_{12} \, \mathbf{t}_2 = \frac{\partial \mathbf{n}}{\partial \ell_1},$$
$$\mathbf{t}_2 \cdot \mathbf{B} = \kappa_{21} \, \mathbf{t}_1 + \kappa_2 \, \mathbf{t}_2 = \frac{\partial \mathbf{n}}{\partial \ell_2}. \tag{6.4.10}$$

These relations demonstrate that

$$\mathbf{P} \cdot \mathbf{B} = \widehat{\boldsymbol{\nabla}} \mathbf{n} \equiv \widehat{\boldsymbol{\nabla}} \otimes \mathbf{n}, \tag{6.4.11}$$

where $\mathbf{P} = \mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2$ is the tangential projection operator and

$$\widehat{\boldsymbol{\nabla}} \equiv \mathbf{P} \cdot \boldsymbol{\nabla} \tag{6.4.12}$$

is the tangential gradient operator. In fact,

$$\mathbf{n} \cdot \widehat{\boldsymbol{\nabla}} \mathbf{n} = \mathbf{n} \cdot \mathbf{P} \cdot \boldsymbol{\nabla} \mathbf{n} = \mathbf{0} \tag{6.4.13}$$

because $\mathbf{n} \cdot \mathbf{P} = \mathbf{0}$, and

$$(\widehat{\boldsymbol{\nabla}} \mathbf{n}) \cdot \mathbf{n} = \mathbf{0} \tag{6.4.14}$$

because of the constant (unit) length of the unit normal vector. Consequently, we may write

$$\mathbf{B} = \mathbf{P} \cdot \boldsymbol{\nabla}\, \mathbf{n} = \widehat{\boldsymbol{\nabla}}\, \mathbf{n}. \tag{6.4.15}$$

This equation defines uniquely the curvature tensor in terms of tangential derivatives of the components of the normal vector.

The Cartesian components of the curvature tensor , indicated by Greek subscripts, are given by

$$B_\alpha = P_{\alpha\gamma} \frac{\partial n_\beta}{\partial x_\gamma}, \tag{6.4.16}$$

where summation is implied over the repeated index, γ.

The mean curvature is given by

$$\kappa_m = \tfrac{1}{2}\, \mathrm{trace}(\mathbf{B}) = \tfrac{1}{2}\, \mathbf{P} : \boldsymbol{\nabla} \mathbf{n}. \tag{6.4.17}$$

In Cartesian coordinates, the double dot product indicated by the colon (:) denotes the sum of the products of all corresponding elements of the matrices on either side.

6.4.5 Regularized curvature tensor

To prevent the occurrence of the zero eigenvalue, we may introduce a regularized curvature tensor defined as

$$\widehat{\mathbf{B}} \equiv \mathbf{B} + \kappa_{\rm n}\, \mathbf{n} \otimes \mathbf{n}, \tag{6.4.18}$$

where $\kappa_{\rm n}$ is an arbitrary constant with dimensions of inverse length. By construction, $\mathbf{t}_{\rm max}$, $\mathbf{t}_{\rm min}$, and $\mathbf{n}$ are eigenvectors of the regularized curvature tensor with corresponding eigenvalues $\kappa_{\rm max}$, $\kappa_{\rm min}$, and $\kappa_{\rm n}$. The Gaussian curvature is given by

$$H = \frac{1}{\kappa_{\rm n}} \det(\widehat{\mathbf{B}}) = \kappa_1 \kappa_2 - \kappa_{12}^2 = \det(\boldsymbol{\kappa}). \tag{6.4.19}$$

As expected, the arbitrary constant κ_n does not appear in the final result.

6.4.6 Numerical evaluation

To evaluate the curvature tensor at a point at a surface, we may consider the variation of the Cartesian components of the position vector, $\mathbf{x}$, and unit normal vector, $\mathbf{n}$, along two generally non-orthogonal surface curvilinear coordinates, ξ and η, and require that

$$\frac{\partial \mathbf{n}}{\partial \xi} = \frac{\partial \mathbf{x}}{\partial \xi} \cdot \mathbf{B}, \qquad \frac{\partial \mathbf{n}}{\partial \eta} = \frac{\partial \mathbf{x}}{\partial \eta} \cdot \mathbf{B}. \tag{6.4.20}$$

Appending to these vector equations the constraint $\mathbf{n}\cdot\mathbf{B} = \mathbf{0}$, we derive three systems of three linear algebraic equations for the three columns of $\mathbf{B}$.

Exercise

6.4.1 Explain why $\mathbf{B} = \kappa_1\, \mathbf{P} + (\kappa_2 - \kappa_1)\, \mathbf{t}_2 \otimes \mathbf{t}_2$, where $\mathbf{P}$ is the tangential projection tensor.

6.5 Curvature tensor in surface coordinates

We may refer to surface curvilinear coordinates and associated surface base vectors, as discussed in Section 6.1, and introduce a representation for the curvature tensor in terms of covariant or contravariant surface base vectors.

6.5.1 Contravariant components

We may introduce the expansion

$$\mathbf{B} = B^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j, \tag{6.5.1}$$

where B^{ij} are the pure contravariant components of the curvature tensor, $\mathbf{B}$. Projecting both sides of this equation from the left onto a contravariant base vector, $\mathbf{g}^m$, where m is a free index, using equation (6.4.15) stating that $\mathbf{B} = \widehat{\boldsymbol{\nabla}}\, \mathbf{n}$, and setting $\mathbf{g}^m \cdot \mathbf{g}_i = \delta_{ij}$, we obtain

$$\mathbf{g}^m \cdot \mathbf{B} = \mathbf{g}^m \cdot \boldsymbol{\nabla}\mathbf{n} = \frac{\partial\, \mathbf{n}}{\partial x_m} = B^{mj}\, \mathbf{g}_j. \tag{6.5.2}$$

Projecting both sides of this vectorial equation onto $\mathbf{g}^n$, where n is a free index, we obtain

$$B^{mn} = \frac{\partial\, \mathbf{n}}{\partial x_m} \cdot \mathbf{g}^n = -\frac{\partial\, \mathbf{g}^n}{\partial x_m} \cdot \mathbf{n}. \tag{6.5.3}$$

The last expression shows that the component of the derivative $\partial \mathbf{g}^n / \partial x_m$ in the direction of the normal vector, $\mathbf{n}$ is $-B^{mn}$.

Equation (6.5.3) can be restated as

$$B^{mn} = -\frac{\partial \mathbf{g}^n}{\partial x_m} \cdot \mathbf{n} = -\frac{\partial^2\, \mathbf{x}}{\partial x_m \partial x_n} \cdot \mathbf{n} = -\frac{\partial\, \mathbf{g}^m}{\partial x_n} \cdot \mathbf{n} = B^{nm}, \tag{6.5.4}$$

which confirms that the matrix of contravariant components of the curvature tensor is symmetric,

$$B^{mn} = B^{nm}. \tag{6.5.5}$$

The symmetry of the pure contravariant components is dictated by the symmetry of the curvature tensor itself, $\mathbf{B}$.

6.5.2 Covariant components

We may also expand

$$\mathbf{B} = B_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j, \tag{6.5.6}$$

where B_{ij} are the covariant components of the curvature tensor, $\mathbf{B}$. Projecting both sides of this expansion from the left onto $\mathbf{g}_m$, where m is a free index, and working as previously in this section, we obtain

$$\mathbf{g}_m \cdot \mathbf{B} = \mathbf{g}_m \cdot \boldsymbol{\nabla} \mathbf{n} = \frac{\partial\, \mathbf{n}}{\partial x^m} = B_{mj}\, \mathbf{g}^j. \tag{6.5.7}$$

Projecting both sides of this equation onto $\mathbf{g}_n$, where n is a free index, we obtain

$$B_{mn} = \frac{\partial\, \mathbf{n}}{\partial x^m} \cdot \mathbf{g}_n = -\frac{\partial\, \mathbf{g}_n}{\partial x^m} \cdot \mathbf{n}. \tag{6.5.8}$$

This expression shows that the component of the derivative $\partial \mathbf{g}_n / \partial x^m$ in the direction of the normal vector, $\mathbf{n}$, is $-B_{mn}$.

Equation (6.5.8) can be restated as

$$B_{mn} = -\frac{\partial\, \mathbf{g}_n}{\partial x^m} \cdot \mathbf{n} = -\frac{\partial^2\, \mathbf{x}}{\partial x^m \partial x^n} \cdot \mathbf{n} = -\frac{\partial\, \mathbf{g}_m}{\partial x^n} \cdot \mathbf{n} = B_{nm}, \tag{6.5.9}$$

which confirms that the matrix of covariant components of the curvature tensor is symmetric,

$$B_{mn} = B_{nm}. \tag{6.5.10}$$

The symmetry of the pure covariant components is dictated by the symmetry of the curvature tensor itself, $\mathbf{B}$.

6.5.3 Mixed components

The curvature tensor can also be expressed in terms on its mixed components, $B_i^{\circ j}$ and $B_{\circ j}^i$ and associated tensor bases,

$$\mathbf{B} = B_{\circ j}^i\, \mathbf{g}_i \otimes \mathbf{g}^j, \qquad \mathbf{B} = B_i^{\circ j}\, \mathbf{g}^i \otimes \mathbf{g}_j. \tag{6.5.11}$$

Working as previously in this section, we find that

$$B_{\circ n}^{m} = \frac{\partial \mathbf{n}}{\partial x_m} \cdot \mathbf{g}_n = -\frac{\partial \, \mathbf{g}_n}{\partial x_m} \cdot \mathbf{n} \tag{6.5.12}$$

and

$$B_{m}^{\circ n} = \frac{\partial \mathbf{n}}{\partial x^m} \cdot \mathbf{g}^n = -\frac{\partial \, \mathbf{g}^n}{\partial x^m} \cdot \mathbf{n}. \tag{6.5.13}$$

These expressions show that $-B_{\circ n}^{m}$ is the component of the derivative $\partial \mathbf{g}_n / \partial x_m$ in the direction of the normal vector, $\mathbf{n}$, and $B_{m}^{\circ n}$ is the component of the derivative $\partial \mathbf{g}^n / \partial x^m$ in the direction of the normal vector, $\mathbf{n}$.

Equation (6.5.12) can be restated as

$$B_{\circ n}^{m} = -\frac{\partial \, \mathbf{g}_n}{\partial x_m} \cdot \mathbf{n} = -\frac{\partial^2 \, \mathbf{x}}{\partial x_m \partial x^n} \cdot \mathbf{n} = -\frac{\partial \, \mathbf{g}^m}{\partial x^n} \cdot \mathbf{n}, \tag{6.5.14}$$

which demonstrates that

$$B_{\circ n}^{m} = B_{n}^{\circ m}. \tag{6.5.15}$$

This property ensures that the curvature tensor is symmetric,

6.5.4 *Mean curvature*

The mean curvature is given by

$$\kappa_m = \tfrac{1}{2} \, \mathrm{trace}(\mathbf{B}) = \tfrac{1}{2} \, B^{ij} g_{ij} = \tfrac{1}{2} \, B_{ij} g^{ij} = \tfrac{1}{2} \, B_{i}^{\circ i} = \tfrac{1}{2} \, B_{\circ i}^{i}, \tag{6.5.16}$$

where summation is implied over repeated indices, i and j.

Using relations (6.1.21) for g^{11}, g^{22}, and $g^{12} = g^{21}$, we find that

$$\kappa_m = \tfrac{1}{2} \, \frac{1}{g} \left(B_{11} g_{22} - 2 \, B_{12} \, g_{12} + B_{22} \, g_{11} \right), \tag{6.5.17}$$

where $g = \det(\mathbf{g}) = g_{11} g_{22} - g_{12}^2$. The right-hand side is defined purely in terms of covariant base vectors. In terms of standard coefficients L, M, and N employed in the literature,

$$L \equiv -B_{11}, \qquad M \equiv -B_{12}, \qquad N \equiv -B_{22}, \tag{6.5.18}$$

expression (6.5.17) for the mean curvature becomes

$$\kappa_m = -\tfrac{1}{2}\,\frac{1}{g}\,\big(\,L\,g_{22} - 2\,M\,g_{12} + N\,g_{11}\,\big). \tag{6.5.19}$$

Denoting

$$E \equiv g_{11}, \qquad F \equiv g_{12} = g_{21}, \qquad G \equiv g_{22}, \tag{6.5.20}$$

we obtain

$$\kappa_m = -\tfrac{1}{2}\,\frac{1}{g}\,\big(\,L\,G - 2\,M\,F + N\,E\,\big), \tag{6.5.21}$$

where $g = EG - F^2$.

6.5.5 Gaussian curvature

The Gaussian curvature is given by

$$H = \frac{1}{\kappa_\mathrm{n}}\,\det(\widehat{\mathbf{B}}) = g\,\det([B^{ij}]) = g\,(B^{11}B^{22} - B^{12}B^{21}), \tag{6.5.22}$$

where $\widehat{\mathbf{B}}$ is the regularized curvature tensor defined in (6.4.18), and $g = \det(\mathbf{g}) = g_{11}g_{22} - g_{12}^2$. Alternative expressions are

$$H = \frac{1}{\kappa_\mathrm{n}}\,\det(\widehat{\mathbf{B}}) = \det([B^{i}_{\circ j}]) = \det([B_{i}^{\circ j}]), \tag{6.5.23}$$

where $\widehat{\mathbf{B}}$ is the regularized curvature tensor defined in (6.4.18), yielding

$$H = B_1^{\circ 1}B_2^{\circ 2} - B_2^{\circ 1}B_1^{\circ 2} = B^1_{\circ 1}B^2_{\circ 2} - B^2_{\circ 1}B^1_{\circ 2}, \tag{6.5.24}$$

The principal curvatures are the eigenvalues of the 2×2 matrix $[B^{i}_{\circ j}]$ or $[B_{i}^{\circ j}]$. A further expression is

$$H = \frac{1}{\kappa_\mathrm{n}}\,\det(\widehat{\mathbf{B}}) = \frac{1}{g}\,\det([B_{ij}]) = \frac{1}{g}\,(B_{11}B_{22} - B_{12}B_{21}), \tag{6.5.25}$$

where $\widehat{\mathbf{B}}$ is the regularized curvature tensor defined in (6.4.18). In terms of the coefficients L, N, M defined in (6.5.18), the Gaussian curvature is given by

$$H = \frac{1}{g}\,(LN - M^2), \tag{6.5.26}$$

where $g = EG - F^2$.

6.5.6 Second fundamental form of a surface

The normal vector changes across the length of an infinitesimal surface vector, $\mathrm{d}\mathbf{x}$ by the differential amount

$$\mathrm{d}\mathbf{n} = \frac{\partial \mathbf{n}}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial \mathbf{n}}{\partial x^2}\,\mathrm{d}x^2. \tag{6.5.27}$$

Projecting this equation onto the corresponding equation

$$\mathrm{d}\mathbf{x} = \mathbf{g}_1\,\mathrm{d}x^1 + \mathbf{g}_2\,\mathrm{d}x^2, \tag{6.5.28}$$

we obtain we obtain the second fundamental form of the surface,

$$\mathcal{S} \equiv -\mathrm{d}\mathbf{n}\cdot\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{n}}{\partial x^i}\cdot\mathbf{g}_j\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{6.5.29}$$

or

$$\mathcal{S} = -B_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{6.5.30}$$

where summation is implied over the repeated indices, i, and j.

Working in a similar fashion, we find that the second fundamental form can be expressed as

$$\mathcal{S} = -B_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = -B^{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j, \tag{6.5.31}$$

and also

$$\mathcal{S} = -B^{i}_{\circ j}\,\mathrm{d}x^i\,\mathrm{d}x_j = -B^{\circ i}_{j}\,\mathrm{d}x_i\,\mathrm{d}x^j. \tag{6.5.32}$$

We observe that the negatives of the components of the curvature tensor are the coefficients of the second fundamental form of the surface in contravariant or covariant coordinates.

6.5.7 Weingarten equation

Equation (6.5.13) states that

$$B^{\circ n}_{m} = \frac{\partial \mathbf{n}}{\partial x^m}\cdot\mathbf{g}^n, \tag{6.5.33}$$

which implies the Weingarten equation

$$\frac{\partial \mathbf{n}}{\partial x^m} = B_m^{\circ n} \mathbf{g}_n. \tag{6.5.34}$$

Note that a normal component is lacking on the right-hand side due to the unit length of the unit normal vector, $\mathbf{n}$. Now we recall that

$$B_m^{\circ n} = B_{mp}\, g^{pn}, \tag{6.5.35}$$

where summation is implied over the repeated index, p, and obtain

$$\frac{\partial \mathbf{n}}{\partial x^m} = B_{mp}\, \mathbf{g}^p. \tag{6.5.36}$$

Using relations (6.1.21) for g^{11}, g^{22}, and $g^{12} = g^{21}$, we find that

$$\begin{aligned}
B_1^{\circ 1} &= B_{11}\, g^{11} + B_{12}\, g^{21} = \frac{B_{11}\, g_{22} - B_{12}\, g^{12}}{g} = \frac{-LG + MF}{g}, \\
B_1^{\circ 2} &= B_{11}\, g^{12} + B_{12}\, g^{22} = \frac{-B_{11}\, g_{21} + B_{12}\, g^{11}}{g} = \frac{LF - ME}{g}, \\
B_2^{\circ 1} &= B_{21}\, g^{11} + B_{22}\, g^{21} = \frac{B_{21}\, g_{22} - B_{22}\, g^{12}}{g} = \frac{-MG + NF}{g}, \\
B_2^{\circ 2} &= B_{21}\, g^{12} + B_{22}\, g^{22} = \frac{-B_{21}\, g_{21} + B_{22}\, g^{11}}{g} = \frac{MF - NE}{g},
\end{aligned} \tag{6.5.37}$$

where $g = EG - F^2$. We recall that the principal curvatures are the eigenvalues of the 2×2 matrix $[B_i^{\circ j}]$ or $[B_{\circ i}^{j}]$.

6.5.8 Weingarten curvature matrix

The Weingarten curvature matrix is defined as

$$\mathbf{W} = -\begin{bmatrix} B_1^{\circ 1} & B_2^{\circ 1} \\ B_1^{\circ 2} & B_2^{\circ 2} \end{bmatrix} = \frac{1}{g} \begin{bmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{bmatrix}. \tag{6.5.38}$$

The principal curvatures are the two eigenvalues of this matrix. Let the corresponding eigenvectors be $\mathbf{w}_1 = [w_{11}, w_{12}]$ and $\mathbf{w}_2 = [w_{21}, w_{22}]$. It can be shown that the following vectors point into the principal directions

$$\mathbf{v}_{\max} = w_{11}\, \mathbf{g}_1 + w_{12}\, \mathbf{g}_2, \qquad \mathbf{v}_{\min} = w_{11}\, \mathbf{g}_1 + w_{12}\, \mathbf{g}_2. \tag{6.5.39}$$

We see that all information needed to construct the curvature tensor is encapsulated in the Weingarten curvature matrix:

$$\mathbf{B} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}, \tag{6.5.40}$$

where the superscript -1 denotes the matrix inverse,

$$\mathbf{V} \equiv \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_{\text{max}} & \mathbf{v}_{\text{min}} & \mathbf{n} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}, \qquad \mathbf{\Lambda} \equiv \begin{bmatrix} \kappa_{\text{max}} & 0 & 0 \\ 0 & \kappa_{\text{min}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.5.41}$$

and $\mathbf{\Lambda}$ the diagonal matrix of principal curvatures.

6.5.9 Curvature over triangles

A surface can be divided into triangles defined by three or six nodes where the unit normal vector is assumed to be known. The curvature tensor, principal curvatures, and principal directions may be evaluated at an arbitrary point over a triangle using the formulas derived in Sections 6.6 and 6.7.

Exercise

6.5.1 Derive expressions (6.5.12) and (6.5.13).

6.6 Curvature over a three-node triangle

A surface can be divided into triangular elements with straight edges defined by three vertices, $\mathbf{x}_i$, for $i = 1, 2, 3$, as illustrated on the left of Figure 6.6.1. Note that the vertices are numbered in the counterclockwise direction around the element contour.

6.6.1 Parametric representation

To describe the element parametrically, we map it to a standard right isosceles triangle in the $\xi\eta$ parametric plane, as shown in Figure 6.6.1. The first element node is mapped to the origin, $\xi = 0, \eta = 0$, the second to the point $\xi = 1, \eta = 0$ on the ξ axis, and the third to the point $\xi = 0, \eta = 1$ on the η axis.

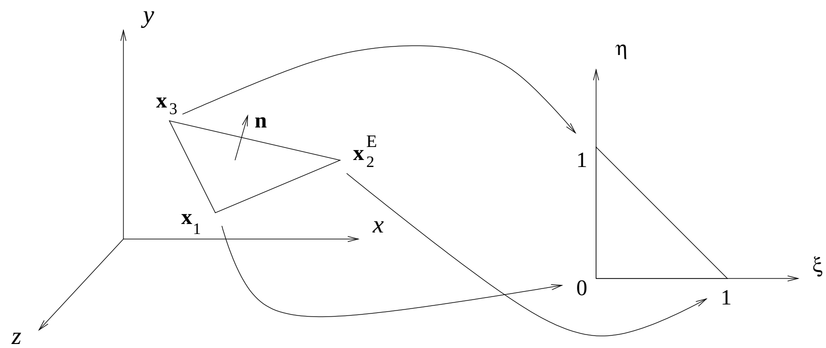

FIGURE 6.6.1 A three-node triangle in three-dimensional space is mapped to a right isosceles triangle in the $\xi\eta$ parametric plane.

The mapping from the physical to the parametric plane is mediated by the linear expansion

$$\mathbf{x} = \mathbf{x}_1\, \psi_1(\xi, \eta) + \mathbf{x}_2\, \psi_2(\xi, \eta) + \mathbf{x}_3\, \psi_3(\xi, \eta), \tag{6.6.1}$$

where $\psi_i(\xi, \eta)$ are element node interpolation functions satisfying a familiar cardinal property: $\psi_i = 1$ at the ith element node and $\psi_i = 0$ at the other two element nodes, so that

$$\psi_i(\xi_j, \eta_j) = \delta_{ij} \tag{6.6.2}$$

for $i, j = 1, 2, 3$, where δ_{ij} is Kronecker's delta, and

$$(\xi_1, \eta_1) = (0, 0), \qquad (\xi_2, \eta_2) = (1, 0), \qquad (\xi_3, \eta_3) = (0, 1) \tag{6.6.3}$$

are the coordinates of the vertices in the $\xi\eta$ plane. We find that

$$\psi_1(\xi, \eta) = \zeta, \qquad \psi_2(\xi, \eta) = \xi, \qquad \psi_3(\xi, \eta) = \eta, \tag{6.6.4}$$

where

$$\zeta = 1 - \xi - \eta. \tag{6.6.5}$$

The trio of variables (ξ, η, ζ) comprise the triangle *barycentric coordinates.* Physically,

$$\zeta = A_1/A, \qquad \xi = A_2/A, \qquad \eta = A_3/A, \tag{6.6.6}$$

where A_1, A_2, and A_3 are the areas of the sub-triangles defined by the field point, $\mathbf{x}$.

Substituting into (6.6.1) the interpolation functions given in (6.6.4), we obtain a mapping function that is a *complete linear function* in ξ and η, consisting of a constant term, a term that is linear in ξ, and a term that is linear in η,

$$\mathbf{x} = \mathbf{x}_1 + (\mathbf{x}_2 - \mathbf{x}_1)\,\xi + (\mathbf{x}_3 - \mathbf{x}_1)\,\eta. \tag{6.6.7}$$

Setting $\xi = 0$ and $\eta = 0$ reveals the first triangle node, $\mathbf{x}_1$.

The unit normal vector is assumed to be provided at the three nodes. Applying the interpolation formula (6.6.7), we obtain

$$\mathbf{n} = \mathbf{n}_1 + (\mathbf{n}_2 - \mathbf{n}_1)\,\xi + (\mathbf{n}_3 - \mathbf{n}_1)\,\eta. \tag{6.6.8}$$

Note that the three nodal normal vectors, $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$, are normal to the surface but not necessarily normal to the flat triangle. In this light, the three-node triangle is regarded as a device for computing the surface curvature in terms of the three nodal normals.

6.6.2 Surface coordinates and curvature

The parameters ξ and η are now regarded as surface contravariant coordinates, so that $x^1 = \xi$ and $x^2 = \eta$. We find that

$$\frac{\partial \mathbf{x}}{\partial \xi} = \mathbf{x}_2 - \mathbf{x}_1, \qquad \frac{\partial \mathbf{x}}{\partial \eta} = \mathbf{x}_3 - \mathbf{x}_1, \tag{6.6.9}$$

and perform a tangential projection to set

$$\mathbf{g}_\xi = \frac{\partial \mathbf{x}}{\partial \xi} \cdot \mathbf{P} = (\mathbf{x}_2 - \mathbf{x}_1) \cdot \mathbf{P}, \qquad \mathbf{g}_\eta = \frac{\partial \mathbf{x}}{\partial \eta} \cdot \mathbf{P} = (\mathbf{x}_3 - \mathbf{x}_1) \cdot \mathbf{P}, \tag{6.6.10}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the tangential projection operator defined in terms of the position dependent normal vector, $\mathbf{n}(\xi, \eta)$. Moreover, we find that

$$\frac{\partial \mathbf{n}}{\partial \xi} = \mathbf{n}_2 - \mathbf{n}_1, \qquad \frac{\partial \mathbf{n}}{\partial \eta} = \mathbf{n}_3 - \mathbf{n}_1 \tag{6.6.11}$$

with a discretization error associated with the linear expansion.

Formulas (6.6.10) and (6.6.11) allow us to compute the two trios of coefficients L, M, N, and E, F, G, involved in the formulas derived in Section 6.5. We recall that

$$E \equiv g_{11}, \qquad F \equiv g_{12} = g_{21}, \qquad G \equiv g_{22}, \tag{6.6.12}$$

and also

$$\begin{aligned} -L \equiv B_{\xi\xi} &= \frac{\partial \mathbf{n}}{\partial \xi} \cdot \mathbf{g}_\xi = (\mathbf{n}_2 - \mathbf{n}_1) \cdot \mathbf{g}_\xi, \\ -M \equiv B_{\xi\eta} &= \frac{\partial \mathbf{n}}{\partial \xi} \cdot \mathbf{g}_\eta = (\mathbf{n}_2 - \mathbf{n}_1) \cdot \mathbf{g}_\eta, \\ -N \equiv B_{\eta\eta} &= \frac{\partial \mathbf{n}}{\partial \eta} \cdot \mathbf{g}_\eta = (\mathbf{n}_3 - \mathbf{n}_1) \cdot \mathbf{g}_\eta. \end{aligned} \tag{6.6.13}$$

The base vectors, $\mathbf{g}_\xi$ and $\mathbf{g}_\eta$, are implicit functions of ξ and η by way of the projection matrix, $\mathbf{P}$. The necessary input includes the positions and unit normal vectors at the location of the three triangle vertices. Having these coefficients available allows us to compute the curvature tensor, mean curvature, and Gaussian curvature using the formulas derived in Section 6.5.

Exercise

6.6.1 Write a code that generates the mean curvature from the positions and unit normal vectors at the location of the three triangle vertices.

6.7 Curvature over a six-node triangle

A surface can be divided into six-node triangles with curved edges defined by three vertex nodes and three edge nodes, as illustrated in Figure 6.7.1. To describe a triangle in parametric form, we map it from the physical xyz space to the familiar right isosceles triangle in the $\xi\eta$ plane, as shown in Figure 6.7.1, as follows:

- The first node is mapped to the origin of the $\xi\eta$ plane, $\xi = 0$, $\eta = 0$.

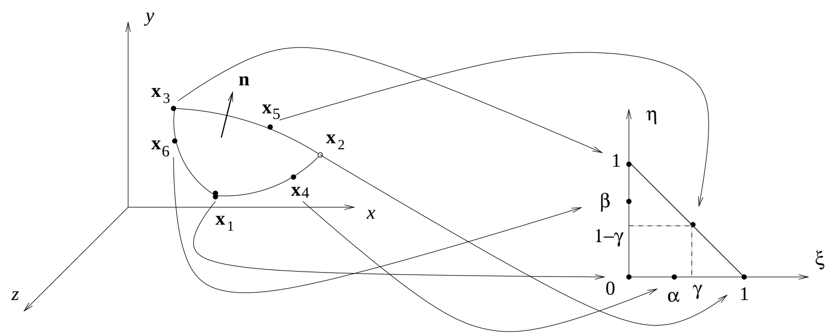

FIGURE 6.7.1 A curved six-node triangle in three-dimensional space is mapped to a flat right isosceles triangle in the $\xi\eta$ parametric plane.

- The second node is mapped to the point $\xi = 1, \eta = 0$ on the ξ axis.
- The third node is mapped to the point $\xi = 0, \eta = 1$ on the η axis.
- The fourth node is mapped to the point $\xi = \alpha, \eta = 0$ on the ξ axis.
- The fifth node is mapped to the point $\xi = \gamma$, $\eta = 1 - \gamma$ on the hypotenuse of the triangle in the $\xi\eta$ plane.
- The sixth node is mapped to the point $\xi = 0, \eta = \beta$ on the η axis.

The dimensionless geometrical mapping coefficients, α, β, and γ are defined as

$$\alpha = \frac{1}{1 + \dfrac{|\mathbf{x}_4 - \mathbf{x}_2|}{|\mathbf{x}_4 - \mathbf{x}_1|}}, \qquad \beta = \frac{1}{1 + \dfrac{|\mathbf{x}_6 - \mathbf{x}_3|}{|\mathbf{x}_6 - \mathbf{x}_1|}},$$
$$\gamma = \frac{1}{1 + \dfrac{|\mathbf{x}_5 - \mathbf{x}_2|}{|\mathbf{x}_5 - \mathbf{x}_3|}}. \qquad (6.7.1)$$

6.7.1 Evaluation of the mapping coefficients

The following Matlab function named *abc*, located in directory TRIANGLE6 of TUNLIB, evaluates these coefficients in terms of the vertex coordinates:

```
function [al,be,ga] = abc ...
...

 (x1,y1,z1 ...
 ,x2,y2,z2 ...
 ,x3,y3,z3 ...
 ,x4,y4,z4 ...
 ,x5,y5,z5 ...
 ,x6,y6,z6 ...
 )

%---------------------------------------
% compute the parametric representation
% coefficients alpha, beta, gamma
%---------------------------------------

 d42 = sqrt( (x4-x2)^2 + (y4-y2)^2 + (z4-z2)^2 );
 d41 = sqrt( (x4-x1)^2 + (y4-y1)^2 + (z4-z1)^2 );
 d63 = sqrt( (x6-x3)^2 + (y6-y3)^2 + (z6-z3)^2 );
 d61 = sqrt( (x6-x1)^2 + (y6-y1)^2 + (z6-z1)^2 );
 d52 = sqrt( (x5-x2)^2 + (y5-y2)^2 + (z5-z2)^2 );
 d53 = sqrt( (x5-x3)^2 + (y5-y3)^2 + (z5-z3)^2 );

 al = 1.0/(1.0+d42/d41);
 be = 1.0/(1.0+d63/d61);
 ga = 1.0/(1.0+d52/d53);

%-----
% done
%-----

 return
```

6.7.2 Interpolation functions

The mapping from the physical to the parametric space is mediated by the vector function

$$\mathbf{x} = \sum_{i=1}^{6} \mathbf{x}_i \, \psi_i(\xi, \eta). \tag{6.7.2}$$

where $\mathbf{x}_i$ are the triangle nodes in three-dimensional space. The quadratic element interpolation functions, $\psi_i(\xi, \eta)$, are required to satisfy cardinal interpolation properties requiring that $\psi_i = 1$ at the ith element node and $\psi_i = 0$ at the other five nodes. In terms of Kronecker's delta, $\delta_{i,j}$,

$$\psi_i(\xi_j, \eta_j) = \delta_{i,j} \tag{6.7.3}$$

for $i, j = 1, \ldots, 6$, where

$$(\xi_1, \eta_1) = (\,0,\, 0\,), \qquad (\xi_2, \eta_2) = (\,1,\, 0\,), \qquad (\xi_3, \eta_3) = (\,0,\, 1\,),$$
$$(\xi_4, \eta_4) = (\,\alpha,\, 0\,), \qquad (\xi_5, \eta_5) = (\,\gamma,\, 1-\gamma\,), \qquad (\xi_6, \eta_6) = (\,0,\, \beta\,) \tag{6.7.4}$$

are the coordinates of the six nodes in the $\xi\eta$ plane.

To derive the ith node interpolation function, we write

$$\psi_i(\xi, \eta) = a_i + b_i\xi + c_i\eta + d_i\xi^2 + e_i\xi\eta + f_i\eta^2, \tag{6.7.5}$$

and compute the six coefficients, a_i–f_i, to satisfy the aforementioned interpolation conditions. The results are shown in Table 6.7.1. The variable $\zeta \equiv 1 - \xi - \eta$ is zero along the hypotenuse where $\eta = 1 - \xi$ and $\xi = 1 - \eta$.

6.7.3 Evaluation of the mean and Gaussian curvature

The parameters ξ and η are now regarded as surface contravariant coordinates, so that $x^1 = \xi$ and $x^2 = \eta$.

The following Matlab function, located in directory TRIANGLE6 of TUNLIB, returns the mean and Gaussian curvatures at a point over the triangle determined by specified values of ξ and η based on formula (6.5.21):

$$\psi_2 = \frac{1}{1-\alpha}\,\xi\,\big(\xi - \alpha + \frac{\alpha-\gamma}{1-\gamma}\,\eta\big)$$
$$\psi_3 = \frac{1}{1-\beta}\,\eta\,\big(\,\eta - \beta + \frac{\beta+\gamma-1}{\gamma}\,\xi\,\big)$$
$$\psi_4 = \frac{1}{\alpha\,(1-\alpha)}\,\xi\,\zeta$$
$$\psi_5 = \frac{1}{\gamma\,(1-\gamma)}\,\xi\,\eta$$
$$\psi_6 = \frac{1}{\beta\,(1-\beta)}\,\eta\,\zeta$$
$$\psi_1 = 1 - \psi_2 - \psi_3 - \psi_4 - \psi_5 - \psi_6$$

TABLE 6.7.1 Element interpolation functions for a six-node triangle, where the variable $\zeta \equiv 1-\xi-\eta$ is the third barycentric coordinate.

```
function [crvm,crvg] = crv6_interp ...
 ...
     (x1,y1,z1 ...
     ,x2,y2,z2 ...
     ,x3,y3,z3 ...
     ,x4,y4,z4 ...
     ,x5,y5,z5 ...
     ,x6,y6,z6 ...
 ...
     ,vx1,vy1,vz1 ...
     ,vx2,vy2,vz2 ...
     ,vx3,vy3,vz3 ...
     ,vx4,vy4,vz4 ...
     ,vx5,vy5,vz5 ...
     ,vx6,vy6,vz6 ...
 ...
     ,al,be,ga ...
     ,xi,eta ...
     )
```

```
%=========================================
% Compute the mean curvature at the nodes
% of a 6-node triangle
%
% x,   y,  z: nodal coordinates
% vx, vy, vz: nodal unit normal vector
%=========================================

%--------
% prepare
%--------

      alc = 1.0-al;
      bec = 1.0-be;
      gac = 1.0-ga;

      alalc = al*alc;
      bebec = be*bec;
      gagac = ga*gac;

%-------------------------------------------
% compute xi derivatives of basis functions
%-------------------------------------------

      dph2 =  (2.0*xi-al+eta*(al-ga)/gac)/alc;
      dph3 =  eta*(be+ga-1.0)/(ga*bec);
      dph4 =  (1.0-2.0*xi-eta)/alalc;
      dph5 =  eta/gagac;
      dph6 = -eta/bebec;
      dph1 = -dph2-dph3-dph4-dph5-dph6;

%-------------------------------------------
% compute eta derivatives of basis functions
%-------------------------------------------

      pph2 =  xi*(al-ga)/(alc*gac);
      pph3 =  (2.0D0*eta-be+xi*(be+ga-1.0D0)/ga)/bec;
      pph4 = -xi/alalc;
```

```
      pph5 =  xi/gagac;
      pph6 =  (1.0D0-xi-2.0D0*eta)/bebec;
      pph1 = -pph2-pph3-pph4-pph5-pph6;

%-------------------------------------
% compute xi and eta derivatives of x
%-------------------------------------

      DxDxi = x1*dph1 + x2*dph2 + x3*dph3 + x4*dph4 ...
            + x5*dph5 + x6*dph6;
      DyDxi = y1*dph1 + y2*dph2 + y3*dph3 + y4*dph4 ...
            + y5*dph5 + y6*dph6;
      DzDxi = z1*dph1 + z2*dph2 + z3*dph3 + z4*dph4 ...
            + z5*dph5 + z6*dph6;

      DxDet = x1*pph1 + x2*pph2 + x3*pph3 + x4*pph4 ...
            + x5*pph5 + x6*pph6;
      DyDet = y1*pph1 + y2*pph2 + y3*pph3 + y4*pph4 ...
            + y5*pph5 + y6*pph6;
      DzDet = z1*pph1 + z2*pph2 + z3*pph3 + z4*pph4 ...
            + z5*pph5 + z6*pph6;

%-------------------------------------
% compute xi and eta derivatives of n
%-------------------------------------

      DvxDxi = vx1*dph1 + vx2*dph2 + vx3*dph3 + vx4*dph4 ...
             + vx5*dph5 + vx6*dph6;
      DvyDxi = vy1*dph1 + vy2*dph2 + vy3*dph3 + vy4*dph4 ...
             + vy5*dph5 + vy6*dph6;
      DvzDxi = vz1*dph1 + vz2*dph2 + vz3*dph3 + vz4*dph4 ...
             + vz5*dph5 + vz6*dph6;

      DvxDet = vx1*pph1 + vx2*pph2 + vx3*pph3 + vx4*pph4 ...
             + vx5*pph5 + vx6*pph6;
      DvyDet = vy1*pph1 + vy2*pph2 + vy3*pph3 + vy4*pph4 ...
             + vy5*pph5 + vy6*pph6;
      DvzDet = vz1*pph1 + vz2*pph2 + vz3*pph3 + vz4*pph4 ...
             + vz5*pph5 + vz6*pph6;
```

```
%----------------------------------------------
% compute the first and second fundamental forms
% of the surface and the mean curvature
%----------------------------------------------

      gxx = DxDxi^2 + DyDxi^2 + DzDxi^2;
      gee = DxDet^2 + DyDet^2 + DzDet^2;
      gxe = DxDxi*DxDet + DyDxi*DyDet + DzDxi*DzDet;

      Bxx = DxDxi*DvxDxi + DyDxi*DvyDxi + DzDxi*DvzDxi;
      Bee = DxDet*DvxDet + DyDet*DvyDet + DzDet*DvzDet;
      Bxe = DxDxi*DvxDet + DyDxi*DvyDet + DzDxi*DvzDet;

      L = -Bxx; M = -Bxe; N = -Bee;
      E = gxx; F = gxe; G = gee;
      g = E*G-F^2;

      crvm = -0.5*(L*G - 2.0*M*F + N*E)/g;
      crvg = (L*N-M^2)/g;

%-----
% done
%-----

 return
```

6.7.4 Triangle on a sphere

The following Matlab code named *crv6*, located in directory TRIANGLE6 of TUNLIB, calls the functions discussed earlier in this section to compute the the mean and Gaussian curvature at a point over the triangle whose vertices lie on the first eighth of a sphere of radius a. for arbitrary values of ξ and η:

```
%---
% vertices on one eighth of a sphere
% of radius 'a'
%---
```

```
a = 3.4; % arbitrary

srt = 1/sqrt(2);
sra = srt*a;

x1 =   a; y1 = 0.0; z1 = 0.0;
x2 = 0.0; y2 =   a; z2 = 0.0;
x3 = 0.0; y3 = 0.0; z3 =   a;
x4 = sra; y4 = sra; z4 = 0.0;
x5 = 0.0; y5 = sra; z5 = sra;
x6 = sra; y6 = 0.0; z6 = sra;

%---
% vertex unit normal (exact)
%---

vx1 = 1.0; vy1 = 0.0; vz1 = 0.0;
vx2 = 0.0; vy2 = 1.0; vz2 = 0.0;
vx3 = 0.0; vy3 = 0.0; vz3 = 1.0;
vx4 = srt; vy4 = srt; vz4 = 0.0;
vx5 = 0.0; vy5 = srt; vz5 = srt;
vx6 = srt; vy6 = 0.0; vz6 = srt;

%---
% compute alpha, beta, gamma
%---

[al,be,ga] = abc ...
...
 (x1,y1,z1 ...
 ,x2,y2,z2 ...
 ,x3,y3,z3 ...
 ,x4,y4,z4 ...
 ,x5,y5,z5 ...
 ,x6,y6,z6 ...
 );

%---
```

```
% compute the curvatures
%---

xi  = 0.21; % example
eta = 0.12; % example

[crvm,crvg] = crv6_interp ...
 ...
  (x1,y1,z1 ...
  ,x2,y2,z2 ...
  ,x3,y3,z3 ...
  ,x4,y4,z4 ...
  ,x5,y5,z5 ...
  ,x6,y6,z6 ...
 ...
  ,vx1,vy1,vz1 ...
  ,vx2,vy2,vz2 ...
  ,vx3,vy3,vz3 ...
  ,vx4,vy4,vz4 ...
  ,vx5,vy5,vz5 ...
  ,vx6,vy6,vz6 ...
 ...
  ,al,be,ga ...
  ,xi,eta ...
  );

format long
[1/a crvm;
1/a^2 crvg]
format short
```

Running the code generates the following output:

```
   0.29411764705882   0.29411764705882
   0.08650519031142   0.08650519031142
```

In this case, the numerical values for the mean and gaussian curvatures are precisely the same as the exact values.

If only the three vertex nodes are given, the three edge nodes and their normals can be computed by interpolation.

Exercise

6.7.1 Use the code discussed in the text to compute the mean and gaussian curvatures at a point on a six-node triangle on the surface of a circular cylinder.

6.8 Curvature tensor and Christoffel symbols

The last expression in (6.5.8) can be rearranged to become

$$-B_{mn} = \frac{\partial\,\mathbf{g}_m}{\partial x^n}\cdot\mathbf{n} \tag{6.8.1}$$

for $m, n = 1, 2$. It is instructive to note the similarity between this relation and the definition of the Christoffel symbols of the second kind given in (4.9.5), repeated below for convenience,

$$\Gamma^k_{mn} \equiv \frac{\partial\,\mathbf{g}_m}{\partial x^n}\cdot\mathbf{g}^k. \tag{6.8.2}$$

In fact, setting $\mathbf{g}^3 = \mathbf{n}$ shows that

$$B_{mn} = -\Gamma^3_{nm} \tag{6.8.3}$$

in the partially orthogonal coordinates defined by $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{n}$. We may write then

$$\frac{\partial\,\mathbf{g}_m}{\partial x^n} \equiv \Gamma^k_{mn}\,\mathbf{g}_k - B_{mn}\,\mathbf{n}, \tag{6.8.4}$$

where the indices vary over 1 and 2. Equation (6.8.4) for surface coordinates is the counterpart of (4.9.3) applicable to volume coordinates.

The last expression in (6.5.13) can be rearranged to give

$$-B^{\circ n}_{m} = \frac{\partial\,\mathbf{g}^n}{\partial x^m}\cdot\mathbf{n}. \tag{6.8.5}$$

We may write then

$$\frac{\partial\,\mathbf{g}^n}{\partial x^m} = -\Gamma^n_{km}\,\mathbf{g}^k - B^{\circ n}_{m}\,\mathbf{n}, \tag{6.8.6}$$

which is the counterpart of (4.9.11) applicable to volume coordinates.

6.8.1 Riemann–Christoffel curvature tensor

Equation (5.8.22), repeated below for convenience,

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2 \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k + \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \frac{\partial \mathbf{g}^k}{\partial x^m}, \tag{6.8.7}$$

provides us with the derivatives of the Christoffel symbols of the second kind. Expressing each one of the derivatives in the last term on the right-hand side in terms of the Christoffel symbols using (6.8.4) and (6.8.6), we obtain

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2 \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k + (\Gamma^n_{ij}\, \mathbf{g}_n - B_{ij}\, \mathbf{n}) \cdot (-\Gamma^k_{pm}\, \mathbf{g}^p - B^{\circ k}_{m}\, \mathbf{n}). \tag{6.8.8}$$

Simplifying the last product, we obtain

$$\frac{\partial \Gamma^k_{ij}}{\partial x^m} = \frac{\partial^2 \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k - \Gamma^n_{ij}\, \Gamma^k_{nm} + B_{ij}\, B^{\circ k}_{m}. \tag{6.8.9}$$

Interchanging the indices j and m, we obtain

$$\frac{\partial \Gamma^k_{im}}{\partial x^j} = \frac{\partial^2 \mathbf{g}_i}{\partial x^m \partial x^j} \cdot \mathbf{g}^k - \Gamma^n_{im}\, \Gamma^k_{nj} + B_{im}\, B^{\circ k}_{j}. \tag{6.8.10}$$

Subtracting the last two equations and rearranging, we obtain

$$\mathcal{R}^k_{\circ imj} = B_{ij} B^{\circ k}_{m} - B_{im} B^{\circ k}_{j}, \tag{6.8.11}$$

where

$$\mathcal{R}^k_{\circ imj} \equiv \frac{\partial \Gamma^k_{ij}}{\partial x^m} - \frac{\partial \Gamma^k_{im}}{\partial x^j} + \Gamma^n_{ij}\Gamma^k_{nm} - \Gamma^n_{im}\Gamma^k_{nj} \tag{6.8.12}$$

are mixed components of the Riemann–Christoffel curvature tensor.

The pure covariant components of the Riemann–Christoffel curvature tensor are given by

$$\mathcal{R}_{kimj} = g_{kp} \mathcal{R}^p_{\circ imj}, \tag{6.8.13}$$

yielding

$$\mathcal{R}_{kimj} = B_{ij} B_{mk} - B_{im} B_{jk}. \tag{6.8.14}$$

We observe that

$$\begin{aligned} \mathcal{R}_{mjki} &= B_{ji} B_{km} - B_{jk} B_{im} = \mathcal{R}_{kimj}, \\ \mathcal{R}_{ikmj} &= B_{kj} B_{mi} - B_{km} B_{ji} = -\mathcal{R}_{kimj}, \\ \mathcal{R}_{kijm} &= B_{im} B_{jk} - B_{ij} B_{mk} = -\mathcal{R}_{kimj}. \end{aligned} \tag{6.8.15}$$

The Riemann–Christoffel curvature tensor discussed in Section 4.12, admits the expansions

$$\begin{aligned} \boldsymbol{\mathcal{R}} &= \mathcal{R}^{k}_{\circ jmi}\, \mathbf{g}_k \otimes \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^m \otimes \mathbf{g}^i \\ &= \mathcal{R}_{kjmi}\, \mathbf{g}^k \otimes \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^m \otimes \mathbf{g}^i. \end{aligned} \tag{6.8.16}$$

Other expansions in terms of mixed or pure contravariant components can be written.

6.8.2 Surface embedded in three-dimensional space

On a surface described by two curvilinear coordinates presently considered, the only non-zero pure covariant components of the Riemann–Christoffel curvature tensor are

$$\mathcal{R}_{1212} = -\mathcal{R}_{2112} = -\mathcal{R}_{1221} = -\mathcal{R}_{2121}. \tag{6.8.17}$$

Direct substitution shows that

$$\mathcal{R}_{1212} = B_{11} B_{22} - B_{12} B_{21} = \det([B_{ij}]). \tag{6.8.18}$$

Using the first expression in (6.5.25), we find that

$$\mathcal{R}_{1212} = gH, \tag{6.8.19}$$

where H is the Gaussian curvature, which is clearly zero on a flat surface.

Exercise

6.8.1 Derive expression (5.8.17).

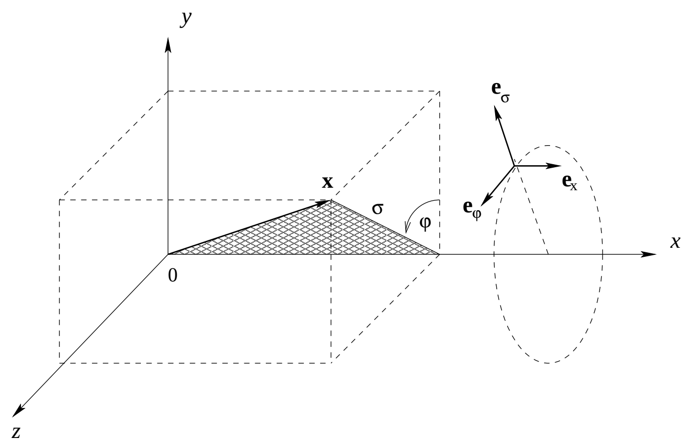

FIGURE 6.9.1 Illustration of cylindrical polar surface coordinates, (φ, x), defined with respect to Cartesian coordinates, (x, y, z).

6.9 Surface of a cylinder

Consider the surface of a cylinder of radius a, and introduce orthogonal cylindrical polar coordinates, (φ, x), where φ is the azimuthal angle, as illustrated in Figure 6.9.1. The doublet, (φ, x), comprise orthogonal curvilinear surface coordinates, where $x^1 = \varphi$ and $x^2 = x$.

The Cartesian coordinates of the position vector on the surface of the cylinder are

$$y = a\cos\varphi, \qquad z = a\sin\varphi. \tag{6.9.1}$$

The base unit vectors are

$$\mathbf{e}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}, \qquad \mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{6.9.2}$$

and the unit normal vector is

$$\mathbf{n} = \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix}. \tag{6.9.3}$$

The corresponding covariant base vectors are given by

$$\mathbf{g}_\varphi = a\,\mathbf{e}_\varphi, \qquad \mathbf{g}_x = \mathbf{e}_x. \tag{6.9.4}$$

The covariant components of the metric tensor are given by

$$g_{\varphi\varphi} = a^2, \qquad g_{\varphi x} = 0, \qquad g_{x\varphi} = 0, \qquad g_{xx} = 1. \tag{6.9.5}$$

The determinant of the matrix consisting of these metric coefficients is $g = a^2$. The surface metric coefficient is $\mathcal{J} = \sqrt{g} = a$.

The surface contravariant base vectors are given by

$$\mathbf{g}^\varphi = \frac{1}{a}\,\mathbf{e}_\varphi, \qquad \mathbf{g}^x = \mathbf{e}_x, \tag{6.9.6}$$

and the contravariant components of the metric tensor are

$$g^{\varphi\varphi} = \frac{1}{a^2}, \qquad g^{\varphi x} = 0, \qquad g^{x\varphi} = 0, \qquad g^{xx} = 1. \tag{6.9.7}$$

Note that $g^{ii} = 1/g_{ii}$, where summation is *not* implied over the repeated index, i.

We find by straightforward differentiation that

$$\frac{\partial\,\mathbf{g}_\varphi}{\partial\varphi} = -a\,\mathbf{n}, \qquad \frac{\partial\,\mathbf{g}_\varphi}{\partial x} = 0,$$
$$\frac{\partial\,\mathbf{g}_x}{\partial\varphi} = 0, \qquad \frac{\partial\,\mathbf{g}_x}{\partial x} = 0. \tag{6.9.8}$$

6.9.1 Christoffel symbols

All surface Christoffel symbols of the second kind turn out to be zero.

6.9.2 Curvature tensor

The covariant components of the surface curvature tensor computed from (6.5.8) are

$$B_{\varphi\varphi} = a, \qquad B_{\varphi x} = B_{x\varphi} = 0, \qquad B_{xx} = 0. \tag{6.9.9}$$

The curvature tensor itself is given by

$$\mathbf{B} = B_{\varphi\varphi}\, \mathbf{g}^{\varphi} \otimes \mathbf{g}^{\varphi}. \tag{6.9.10}$$

Making substitutions, we obtain

$$\mathbf{B} = \frac{1}{a}\, \mathbf{e}_{\varphi} \otimes \mathbf{e}_{\varphi}. \tag{6.9.11}$$

The mean curvature is

$$\kappa_m = \frac{1}{2a}. \tag{6.9.12}$$

Exercise

6.9.1 Compute the principal curvatures and the Gaussian curvature of a cylindrical surface.

6.10 Surface of a sphere

Consider the surface of a sphere of radius a, and introduce orthogonal spherical polar coordinates where $x^1 = \theta$ is the meridional angle and $x^2 = \varphi$ is the azimuthal angle, as shown in Figure 6.10.1. The doublet, (θ, φ), comprise orthogonal curvilinear surface coordinates.

The Cartesian coordinates of the position vector on the surface of the sphere are

$$x = a\cos\theta, \qquad y = a\sin\theta\cos\varphi, \qquad z = a\sin\theta\sin\varphi. \tag{6.10.1}$$

The base unit vectors are

$$\mathbf{e}_{\theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \end{bmatrix}, \qquad \mathbf{e}_{\varphi} = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}, \tag{6.10.2}$$

and the unit normal vector is

$$\mathbf{n} = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix}. \tag{6.10.3}$$

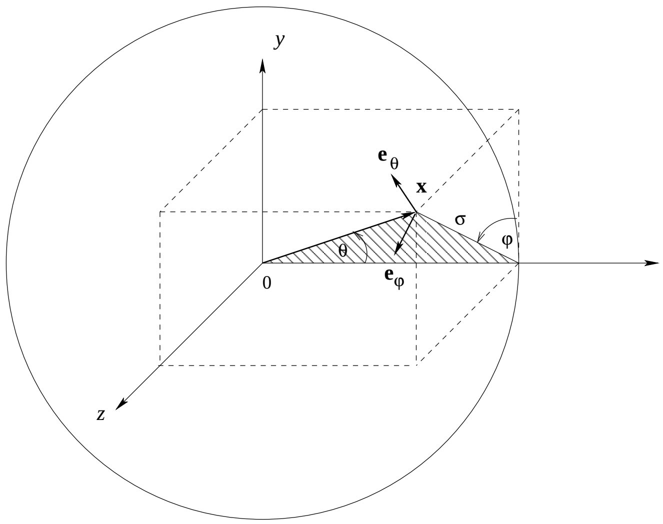

FIGURE 6.10.1 Illustration of spherical polar surface coordinates, (θ, φ), defined with respect to the Cartesian coordinates, (x, y, z) where θ is the meridional angle and φ is the azimuthal angle.

The corresponding covariant base vectors are

$$\mathbf{g}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = a\,\mathbf{e}_\theta, \qquad \mathbf{g}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = a\,\sin\theta\,\mathbf{e}_\varphi, \tag{6.10.4}$$

and the covariant components of the metric tensor are

$$\begin{aligned} g_{\theta\theta} &= a^2, \qquad g_{\theta\varphi} = 0, \\ g_{\varphi\theta} &= 0, \qquad g_{\varphi\varphi} = a^2 \sin^2\theta. \end{aligned} \tag{6.10.5}$$

The determinant of the matrix consisting of these metric coefficients is given by

$$g = a^4 \sin^2\theta \tag{6.10.6}$$

and the surface metric coefficient is

$$\mathcal{J} = \sqrt{g} = a^2 \sin\theta \tag{6.10.7}$$

for $0 \leq \theta \leq \pi$.

The surface contravariant base vectors are

$$\mathbf{g}^\theta = \frac{1}{a}\,\mathbf{e}_\theta, \qquad \mathbf{g}^\varphi = \frac{1}{a\sin\theta}\,\mathbf{e}_\varphi, \tag{6.10.8}$$

and the contravariant components of the metric tensor are

$$g^{\theta\theta} = \frac{1}{a^2}, \qquad g^{\theta\varphi} = 0, \quad g^{\varphi\theta} = 0, \quad g^{\varphi\varphi} = \frac{1}{a^2\sin^2\theta}. \tag{6.10.9}$$

Note that $g^{ii} = 1/g_{ii}$, where summation is *not* implied over the repeated index, i.

We find by straightforward differentiation that

$$\frac{\partial\,\mathbf{g}_\theta}{\partial\theta} = -a\,\mathbf{n}, \qquad \frac{\partial\,\mathbf{g}_\theta}{\partial\varphi} = a\,\cos\theta\,\mathbf{e}_\varphi. \tag{6.10.10}$$

Moreover, we find that

$$\frac{\partial\,\mathbf{g}_\varphi}{\partial\varphi} = -a\,\sin\theta \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix}, \qquad \frac{\partial\,\mathbf{g}_\varphi}{\partial\theta} = a\,\cos\theta\,\mathbf{e}_\varphi. \tag{6.10.11}$$

6.10.1 Christoffel symbols

Using expression (4.9.5) for the Christoffel symbols of the second kind, we find that

$$\Gamma^\theta_{\varphi\varphi} = \frac{\partial\mathbf{g}_\varphi}{\partial\varphi}\cdot\mathbf{g}^\theta \tag{6.10.12}$$

and then

$$\Gamma^\theta_{\varphi\varphi} = a\,\sin\theta \begin{bmatrix} 0 \\ -\cos\varphi \\ -\sin\varphi \end{bmatrix} \cdot \frac{1}{a} \begin{bmatrix} -\sin\theta \\ \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \end{bmatrix}. \tag{6.10.13}$$

Carrying out the multiplications, we obtain

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta. \tag{6.10.14}$$

Working in a similar fashion, we find that

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{\partial \mathbf{g}_\theta}{\partial \varphi} \cdot \mathbf{g}^{\varphi} \tag{6.10.15}$$

and then

$$\Gamma^{\varphi}_{\theta\varphi} = a \begin{bmatrix} 0 \\ -\cos\theta \sin\varphi \\ \cos\theta\cos\varphi \end{bmatrix} \cdot \frac{1}{a\sin\theta} \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}. \tag{6.10.16}$$

Carrying out the multiplications, we obtain

$$\Gamma^{\varphi}_{\theta\varphi} = \cot\theta. \tag{6.10.17}$$

Working in a similar fashion, we find that

$$\Gamma^{\varphi}_{\varphi\theta} = \frac{\partial \mathbf{g}_\varphi}{\partial \theta} \cdot \mathbf{g}^{\varphi} \tag{6.10.18}$$

and then

$$\Gamma^{\varphi}_{\varphi\theta} = a\,\cos\theta \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix} \cdot \frac{1}{a\sin\theta} \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}. \tag{6.10.19}$$

Carrying out the multiplications, we obtain

$$\Gamma^{\varphi}_{\varphi\theta} = \cot\theta. \tag{6.10.20}$$

All other Christoffel symbols of the second kind turn out to be zero.

6.10.2 Curvature tensor

The covariant components of the curvature tensor computed from (6.5.8) are

$$B_{\theta\theta} = a, \qquad B_{\theta\varphi} = B_{\varphi\theta} = 0, \qquad B_{\varphi\varphi} = a\sin^2\theta. \tag{6.10.21}$$

The curvature tensor itself is given by

$$\mathbf{B} = B_{\theta\theta}\,\mathbf{g}^{\theta} \otimes \mathbf{g}^{\theta} + B_{\varphi\varphi}\,\mathbf{g}^{\varphi} \otimes \mathbf{g}^{\varphi}. \tag{6.10.22}$$

Making substitutions, we obtain

$$\mathbf{B} = \frac{1}{a}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi\right) = \frac{1}{a}\,\mathbf{P}, \tag{6.10.23}$$

where $\mathbf{P}$ is the tangential projection tensor reflecting the isotropy of the spherical shape. The mean curvature is $\kappa_m = \frac{1}{a}$.

Exercise

6.10.1 Compute the contravariant components of the curvature tensor over a sphere in the (θ, φ) surface coordinates.

6.11 Surface divergence of a vector field

A vector field, $\mathbf{u}$, with tangential and normal components defined over a surface can be expressed as

$$\mathbf{u} = u^i\,\mathbf{g}_i + u_n\,\mathbf{n} = u_i\,\mathbf{g}^i + u_n\,\mathbf{n}, \tag{6.11.1}$$

where summation is implied over the repeated index $i = 1, 2$, $\mathbf{g}_i$ are covariant surface base vectors, u^i are contravariant surface components, $\mathbf{g}^i$ are covariant surface base vectors, u_i are covariant surface components, u_n is the normal component of $\mathbf{u}$, and $\mathbf{n}$ is the unit normal vector.

The surface divergence of $\mathbf{u}$ is a scalar given by

$$\varrho \equiv \widehat{\boldsymbol{\nabla}} \cdot \mathbf{u}, \tag{6.11.2}$$

where $\widehat{\boldsymbol{\nabla}} = \mathbf{P}\cdot\boldsymbol{\nabla}$ is the tangential gradient operator and $\mathbf{P} = \mathbf{I} - \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}$ is the tangential projection operator, as discussed in Section 8.2.

In index notation,

$$\varrho = P_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} = \frac{\partial u_\alpha}{\partial x_\alpha} - n_\alpha n_\beta \frac{\partial u_\alpha}{\partial x_\beta}, \tag{6.11.3}$$

where summation is implied over Greek indices denoting Cartesian coordinates, $\alpha, \beta = 1, 2.3$. For example, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Using expressions (5.2.1) for the gradient operator restricted to surface coordinates, we set

$$\varrho = \mathbf{g}^k \cdot \frac{\partial}{\partial x^k}(u^i\,\mathbf{g}_i + u_n\mathbf{n}) = \mathbf{g}^k \cdot \frac{\partial}{\partial x^k}(u_i\,\mathbf{g}^i + u_n\mathbf{n}), \qquad (6.11.4)$$

where summation is implied over the repeated indices i and k in the range $i, k = 1, 2$.

Expanding the derivative of the expression between the two equal sings in (6.11.4), and noting that $\mathbf{g}^k \cdot \mathbf{n} = 0$, we obtain

$$\varrho = \mathbf{g}^k \cdot \mathbf{g}_i\,\frac{\partial u^i}{\partial x^k} + \mathbf{g}^k \cdot \frac{\partial\,\mathbf{g}_i}{\partial x^k}\,u^i + \mathbf{g}^k \cdot \frac{\partial \mathbf{n}}{\partial x^k}\,u_n. \qquad (6.11.5)$$

Now setting $\mathbf{g}^k \cdot \mathbf{g}_i = \delta_{ij}$ in the first term on the right-hand side, and expressing the last term in terms of the curvature tensor, $\mathbf{B}$, we obtain

$$\varrho = \frac{\partial u^i}{\partial x^i} + \mathbf{g}^k \cdot \frac{\partial \mathbf{g}_i}{\partial x^k}\,u^i + \mathbf{g}^k \cdot \mathbf{g}^j\,B_{kj}u_n. \qquad (6.11.6)$$

Both the tangential and normal components of $\mathbf{u}$ are involved in this equation.

6.11.1 Normal motion and mean curvature

Next, we invoke relation (6.8.4) to express the derivative in the second term on the right-hand side of (6.11.6) in terms of the Christoffel symbols of the second kind and the curvature tensor,

$$\frac{\partial\,\mathbf{g}_i}{\partial x^k} = \Gamma^m_{ik}\,\mathbf{g}_m - B_{ik}\,\mathbf{n} \qquad (6.11.7)$$

for $i, k, m = 1, 2$. The final result is

$$\varrho = \frac{\partial u^i}{\partial x^i} + \Gamma^k_{ik}\,u^i + g^{kj}\,B_{kj}\,u_n. \qquad (6.11.8)$$

Now referring to (6.5.16), we set $g^{kj}B_{kj} = 2\kappa_m$ and obtain

$$\varrho = \frac{\partial u^i}{\partial x^i} + \Gamma^k_{ik}\,u^i + 2\,\kappa_m\,u_n, \qquad (6.11.9)$$

where κ_m is the mean curvature. We see that the normal components, u_n, contributes to the surface divergence only in the case of non-zero mean curvature. This contribution arises in the case of a spherical interface expanding or contracting with normal velocity u_n. In terms of the covariant derivative defined in (4.13.5), equation (6.11.9) the simple form

$$\varrho = u^i_{,i} + 2\,\kappa_m\,u_n, \tag{6.11.10}$$

where

$$u^i_{,i} = \frac{\partial u^i}{\partial x^i} + \Gamma^k_{ik}\,u^i \tag{6.11.11}$$

and summation is implied over the repeated indices, $i, k = 1, 2$.

6.11.2 Cylindrical surface

In the case of a cylindrical surface of radius a, we introduce orthogonal cylindrical surface coordinates, φ and x, as discussed in Section 6.9. Recalling that all Christoffel symbols are zero, and setting $\kappa_m = 1/(2a)$, we obtain from (6.11.9) the simplified expression

$$\varrho = \frac{\partial u^\varphi}{\partial \varphi} + \frac{\partial u^x}{\partial x} + \frac{1}{a}\,u_n, \tag{6.11.12}$$

where u^φ and u^x are contravariant components.

We may introduce physical vector components indicated by a caret associated with the dimensionless unit vectors $\mathbf{e}_\varphi$ and $\mathbf{e}_x$, and expand the vector field $\mathbf{u}$ as

$$\mathbf{u} = \hat{u}^\varphi\,\mathbf{e}_\varphi + \hat{u}^x\,\mathbf{e}_x + \hat{u}_n\,\mathbf{n}. \tag{6.11.13}$$

Since $\mathbf{g}_\varphi = a\,\mathbf{e}_\varphi$ and $\mathbf{g}_x = \mathbf{e}_x$, we find that

$$u^\varphi = \frac{1}{a}\,\hat{u}^\varphi, \qquad u^x = \hat{u}^x, \qquad u^n = \hat{u}^n. \tag{6.11.14}$$

Substituting these expressions into (6.11.12), we obtain

$$\varrho = \frac{1}{a}\,\frac{\partial \hat{u}^\varphi}{\partial \varphi} + \frac{\partial \hat{u}^x}{\partial x} + \frac{1}{a}\,\hat{u}_n. \tag{6.11.15}$$

Only the last term survives in the case of a cylindrical surface expanding with normal velocity u_n.

6.11.3 Spherical surface

In the case of a spherical surface of radius a, we introduce orthogonal surface contravariant coordinates θ and φ, as discussed in Section 6.10, and obtain

$$\varrho = \frac{\partial u^\theta}{\partial \theta} + \frac{\partial u^\varphi}{\partial \varphi} + \Gamma^\varphi_{\theta\varphi}\, u^\theta + \frac{2}{a}\, u_n. \tag{6.11.16}$$

Substituting the expressions for the Christoffel symbols, we find that

$$\varrho = \frac{\partial u^\theta}{\partial \theta} + \frac{\partial u^\varphi}{\partial \varphi} + \cot\theta\, u^\theta + \frac{2}{a}\, u_n, \tag{6.11.17}$$

which can be rearranged into

$$\varrho = \frac{1}{\sin\theta}\,\frac{\partial(\sin\theta\, u^\theta)}{\partial \theta} + \frac{\partial u^\varphi}{\partial \varphi} + \frac{2}{a}\, u_n, \tag{6.11.18}$$

where u^θ and u^φ are contravariant components.

We may introduce physical vector components indicated by a caret associated with the dimensionless unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$, and expand the vector field $\mathbf{u}$ as

$$\mathbf{u} = \hat{u}^\theta\, \mathbf{e}_\theta + \hat{u}^\varphi\, \mathbf{e}_\varphi + \hat{u}_n\, \mathbf{n}. \tag{6.11.19}$$

Since $\mathbf{g}_\theta = a\,\mathbf{e}_\theta$ and $\mathbf{g}_\varphi = a\sin\theta\,\mathbf{e}_\varphi$, we find that

$$u^\theta = \frac{1}{a}\,\hat{u}^\theta, \qquad u^\varphi = \frac{1}{a\sin\theta}\,\hat{u}^\varphi, \qquad u^n = \hat{u}^n. \tag{6.11.20}$$

Substituting these expressions into (6.11.17), we obtain

$$\varrho = \frac{1}{a}\,\frac{\partial \hat{u}^\theta}{\partial \theta} + \frac{1}{a\sin\theta}\,\frac{\partial \hat{u}^\varphi}{\partial \varphi} + \frac{\cot\theta}{a}\,\hat{u}^\theta + \frac{2}{a}\,\hat{u}_n, \tag{6.11.21}$$

which can be rearranged into

$$\varrho = \frac{1}{a\sin\theta}\Big(\frac{\partial(\sin\theta\,\hat{u}^\theta)}{\partial \theta} + \frac{\partial \hat{u}^\varphi}{\partial \varphi}\Big) + \frac{2}{a}\,\hat{u}_n. \tag{6.11.22}$$

Only the last term survives in the case of a spherical surface expanding with normal velocity u_n.

Exercise

6.11.1 Write equation (6.11.10) for a flat interface.

6.12 Surface gradient of a vector field

The surface gradient of a surface vector field, $\mathbf{u}$, possessing tangential and normal components, is a two-index tensor given by

$$\mathbf{L} \equiv \widehat{\boldsymbol{\nabla}}\, \mathbf{u} \equiv \widehat{\boldsymbol{\nabla}} \otimes \mathbf{u}. \tag{6.12.1}$$

In index notation, the Cartesian components of $\mathbf{L}$ are given by

$$L_{\alpha\beta} = P_{\alpha\gamma} \frac{\partial u_\beta}{\partial x_\gamma} = \frac{\partial u_\beta}{\partial x_\alpha} - n_\alpha n_\gamma \frac{\partial u_\beta}{\partial x_\gamma}, \tag{6.12.2}$$

where $\mathbf{P}$ is the surface projection tensor and summation is implied over Greek indices denoting Cartesian coordinates. For example, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Using expressions (5.2.1) for the gradient operator restricted to surface coordinates, we set

$$\mathbf{L} = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(u^i\, \mathbf{g}_i + u_n \mathbf{n}) = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(u_i\, \mathbf{g}^i + u_n \mathbf{n}). \tag{6.12.3}$$

Expanding the derivative of the expression enclosed by parentheses after the first equal sing, we obtain

$$\mathbf{L} = \mathbf{g}^k \otimes \mathbf{g}_i \frac{\partial u^i}{\partial x^k} + \mathbf{g}^k \otimes \frac{\partial\, \mathbf{g}_i}{\partial x^k}\, u^i + \mathbf{g}^k \otimes \frac{\partial \mathbf{n}}{\partial x^k}\, u_n + \mathbf{g}^k \otimes \mathbf{n}\, \frac{\partial u_n}{\partial x^k}. \tag{6.12.4}$$

Expressing the penultimate term on the right-hand side in terms of the curvature tensor, we obtain

$$\mathbf{L} = \mathbf{g}^k \otimes \mathbf{g}_i \frac{\partial u^i}{\partial x^k} + \mathbf{g}^k \otimes \frac{\partial \mathbf{g}_i}{\partial x^k} u^i + \mathbf{g}^k \otimes \mathbf{g}^j\, B_{kj} u_n + \mathbf{g}^k \otimes \mathbf{n}\, \frac{\partial u_n}{\partial x^k}. \tag{6.12.5}$$

Next, we express the derivative $\partial \mathbf{g}_i / \partial x^k$ on the right-hand side in terms of the Christoffel symbols of the second kind and the curvature tensor using (6.8.4), repeated below with redefined indices for convenience,

$$\frac{\partial \mathbf{g}_i}{\partial x^k} = \Gamma^m_{ik} \, \mathbf{g}_m - B_{ik} \, \mathbf{n} \tag{6.12.6}$$

for $i, k, m = 1, 2$, and obtain

$$\mathbf{L} = \mathbf{g}^k \otimes \mathbf{g}_i \, \frac{\partial u^i}{\partial x^k} + \mathbf{g}^k \otimes \mathbf{g}_m \Gamma^m_{ik} \, u^i - \mathbf{g}^k \otimes \mathbf{n} \, B_{ik} \, u^i \\ + \mathbf{g}^k \otimes \mathbf{g}^j B_{kj} u_n + \mathbf{g}^k \otimes \mathbf{n} \, \frac{\partial u_n}{\partial x^k}. \tag{6.12.7}$$

Rearranging, we obtain

$$\mathbf{L} = \mathbf{g}^k \otimes \mathbf{g}_m \left(\frac{\partial u^m}{\partial x^k} + \Gamma^m_{ik} \, u^i \right) \\ + \mathbf{g}^k \otimes \mathbf{g}^j B_{kj} u_n + \mathbf{g}^k \otimes \mathbf{n} \left(\frac{\partial u_n}{\partial x^k} - B_{ik} \, u^i \right). \tag{6.12.8}$$

Expressing the term inside the first set of parentheses in terms of the covariant derivative of a contravariant component, $u^m_{,k}$, we obtain the compact form

$$\mathbf{L} = \mathbf{g}^k \otimes \mathbf{g}_m \, u^m_{,k} + \mathbf{g}^k \otimes \mathbf{g}^j \, B_{kj} u_n \\ + \mathbf{g}^k \otimes \mathbf{n} \left(\frac{\partial u_n}{\partial x^k} - B_{ik} \, u^i \right). \tag{6.12.9}$$

The trace of $\mathbf{L}$ is the surface divergence of $\mathbf{u}$,

$$\varrho = \text{trace}(\mathbf{L}) = \delta_{km} \, u^m_{,k} + g^{kj} \, B_{kj} u_n, \tag{6.12.10}$$

as derived previously in (6.11.10).

Exercise

6.12.1 Derive expression (6.12.10) from (6.12.9).

6.13 Surface divergence of a surface tensor field

A tangential surface tensor field, $\mathbf{T}$, can be expressed in terms of surface base vectors in four combinations,

$$\begin{aligned}\mathbf{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T_i^{\circ j}\mathbf{g}^i \otimes \mathbf{g}_j \\ = T^i_{\circ j}\,\mathbf{g}_i \otimes \mathbf{g}^j = T_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j,\end{aligned} \tag{6.13.1}$$

where $\mathbf{g}_i$ are covariant and $\mathbf{g}^i$ are contravariant surface base vectors. The stipulated absence of normal components on the right-hand side of (6.13.1) implies that

$$\mathbf{n} \cdot \mathbf{T} = \mathbf{0}, \qquad \mathbf{T} \cdot \mathbf{n} = \mathbf{0}. \tag{6.13.2}$$

These restrictions can be lifted by straightforward modifications.

6.13.1 Notational inconsistency

An unfortunate notational inconsistency has been introduced to conform with standard convention. The ijth element of the tensor $\mathbf{T}$ is denoted as T_{ij}, and the ij covariant component of the same tensor has also been denoted as T_{ij}. For clarity, the former could have been denoted as $T(i,j)$; alternatively, Greek indices can be employed.

6.13.2 Surface divergence

The surface divergence of $\mathbf{T}$ is a surface vector field possessing tangential and normal components, given by

$$\boldsymbol{\psi} \equiv \widehat{\boldsymbol{\nabla}} \cdot \mathbf{T}, \tag{6.13.3}$$

where $\widehat{\boldsymbol{\nabla}} = \mathbf{P}\cdot\boldsymbol{\nabla}$ is the tangential gradient operator. In index notation, the Cartesian components of $\boldsymbol{\psi}$ are given by

$$\psi_\gamma = P_{\alpha\beta}\,\frac{\partial T_{\alpha\gamma}}{\partial x_\beta}, \tag{6.13.4}$$

where summation is implied over Greek indices denoting Cartesian coordinates. For example, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

6.13.3 Resolution into tangential and normal components

Resolving $\boldsymbol{\psi}$ into tangential and normal components, we obtain

$$\boldsymbol{\psi} = \psi^j \mathbf{g}_j + \psi_n \mathbf{n} = \psi_j \mathbf{g}^j + \psi_n \mathbf{n}, \tag{6.13.5}$$

where ψ^j are the contravariant surface components, ψ_j are the covariant surface components, ψ_n is the normal component, and $\mathbf{n}$ is the unit normal vector.

Using expressions (5.2.1) for the gradient operator restricted to surface coordinates, we obtain

$$\boldsymbol{\psi} = \mathbf{g}^k \cdot \frac{\partial}{\partial x^k}(T^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j) = \mathbf{g}^k \cdot \frac{\partial}{\partial x^k}(T_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j), \tag{6.13.6}$$

where summation is implied over the repeated indices, i, j, and k. Four similar expressions can be written involving the mixed components of $\mathbf{T}$.

Expanding the derivative in the first expression of (6.13.6), we obtain the expression

$$\boldsymbol{\psi} = \frac{\partial T^{ij}}{\partial x^k}\, \mathbf{g}^k \cdot \mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\, \mathbf{g}^k \cdot \frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\, \mathbf{g}^k \cdot \mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}, \tag{6.13.7}$$

where $\mathbf{g}^k \cdot \mathbf{g}_i = \delta_{ki}$ in the first and last terms on the right-hand side.

Next, we use the properties of Kronecker's delta and express the last two derivatives of the covariant base vectors on the right-hand side of (6.13.7) in terms of the Christoffel symbols of the second kind and the curvature tensor using (6.8.4), repeated below with renamed indices for convenience,

$$\frac{\partial\, \mathbf{g}_i}{\partial x^k} \equiv \Gamma^m_{ik}\, \mathbf{g}_m - B_{ik}\, \mathbf{n}. \tag{6.13.8}$$

Rearranging, we obtain

$$\boldsymbol{\psi} = \frac{\partial T^{ij}}{\partial x^i}\, \mathbf{g}_j + T^{ij}\, \Gamma^k_{ik}\, \mathbf{g}_j + T^{ij}\, \Gamma^m_{ji}\, \mathbf{g}_m - T^{ij} B_{ji}\, \mathbf{n}. \tag{6.13.9}$$

Mutually renaming the indices j and m in the penultimate term on the right-hand side, we obtain

$$\boldsymbol{\psi} = \frac{\partial T^{ij}}{\partial x^i}\,\mathbf{g}_j + T^{ij}\,\Gamma^k_{ik}\,\mathbf{g}_j + T^{im}\,\Gamma^m_{mi}\,\mathbf{g}_j - T^{ij} B_{ji}\,\mathbf{n}. \quad (6.13.10)$$

We have found that

$$\boldsymbol{\psi} = \psi^j\,\mathbf{g}_j - T^{ij} B_{ji}\,\mathbf{n}, \quad (6.13.11)$$

where

$$\psi^j = \frac{\partial T^{ij}}{\partial x^i} + \Gamma^k_{mk}\,T^{mj} + \Gamma^j_{mi}\,T^{im}. \quad (6.13.12)$$

are the contravariant components of $\boldsymbol{\psi}$. In abbreviated notation,

$$\psi^j = T^{ij}_{,i}, \quad (6.13.13)$$

where the general covariant derivative $T^{ij}_{,k}$ is defined in (4.14.7) as

$$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\,T^{mj} + \Gamma^j_{mk}\,T^{im}. \quad (6.13.14)$$

In summary, the surface divergence of a surface tensor field is given by

$$\boldsymbol{\psi} \equiv (\mathbf{P}\cdot\boldsymbol{\nabla})\cdot\mathbf{T} = T^{ij}_{,i}\,\mathbf{g}_j - T^{ij}\,B_{ji}\,\mathbf{n}. \quad (6.13.15)$$

This expression appears in equilibrium equations governing the shapes of membranes and shells, as discussed in Sections 6.6–6.8.

Exercise

6.13.1 Derive an expression for the surface divergence over a sphere.

6.14 Surface gradient of a surface tensor field

A surface tensor field, $\mathbf{T}$, can be expressed in terms of covariant or contravariant surface base vectors as

$$\mathbf{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T_i^{\circ j}\mathbf{g}^i \otimes \mathbf{g}_j$$
$$= T^i_{\circ j}\,\mathbf{g}_i \otimes \mathbf{g}^j = T_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j. \tag{6.14.1}$$

The surface gradient of $\mathbf{T}$ is a three-index tensor given by

$$\mathbf{N} \equiv \widehat{\boldsymbol{\nabla}}\,\mathbf{T} \equiv \widehat{\boldsymbol{\nabla}} \otimes \mathbf{T}, \tag{6.14.2}$$

where $\widehat{\boldsymbol{\nabla}} = \mathbf{P}\cdot\boldsymbol{\nabla}$ is the tangential gradient operator. In index notation, the Cartesian components of $\mathbf{N}$ are given by

$$N_{\alpha\beta\gamma} = P_{\alpha\delta}\,\frac{\partial T_{\beta\gamma}}{\partial x_\delta} = \frac{\partial T_{\beta\gamma}}{\partial x_\alpha} - n_\alpha n_\delta \frac{\partial T_{\beta\gamma}}{\partial x_\delta}, \tag{6.14.3}$$

where summation is implied over Greek indices denoting Cartesian coordinates. For example, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

6.14.1 Surface gradient in terms of the curvature

Using expressions (5.2.1) for the gradient operator in surface coordinates and the first expansion in (6.14.1), we set

$$\mathbf{N} = \mathbf{g}^k \otimes \frac{\partial}{\partial x^k}(T^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j). \tag{6.14.4}$$

Expanding the derivatives, we obtain

$$\mathbf{N} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j$$
$$+T^{ij}\mathbf{g}^k \otimes \frac{\partial\,\mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}^k \otimes \mathbf{g}_i \otimes \frac{\partial\,\mathbf{g}_j}{\partial x^k}. \tag{6.14.5}$$

Next, we express the derivatives $\partial\,\mathbf{g}_i/\partial x^k$ and $\partial\,\mathbf{g}_j/\partial x^k$ on the right-hand side in terms of the Christoffel symbols of the second kind and the curvature tensor using (6.8.4), repeated below with redefined indices for convenience,

$$\frac{\partial\,\mathbf{g}_i}{\partial x^k} = \Gamma^m_{ik}\,\mathbf{g}_m - B_{ik}\,\mathbf{n}. \tag{6.14.6}$$

Rearranging, we obtain

$$\begin{aligned}\mathbf{N} = \frac{\partial T^{ij}}{\partial x^k}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j\, + \Gamma^m_{ik}\, T^{ij}\mathbf{g}^k \otimes \mathbf{g}_m \otimes \mathbf{g}_j \\ -B_{ik}\, T^{ij}\mathbf{g}^k \otimes \mathbf{n} \otimes \mathbf{g}_j + \Gamma^m_{jk}\, T^{ij}\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_m \\ -B_{jk}\, T^{ij}\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{n}.\end{aligned} \tag{6.14.7}$$

Rearranging further, we obtain the final expression

$$\begin{aligned}\mathbf{N} = \Big(\frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\, T^{mj} + \Gamma^j_{mk}\, T^{im}\Big)\, \mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{g}_j \\ -T^{ij}\,\big(\, B_{ik}\,\mathbf{g}^k \otimes \mathbf{n} \otimes \mathbf{g}_j + B_{jk}\,\mathbf{g}^k \otimes \mathbf{g}_i \otimes \mathbf{n}\big).\end{aligned} \tag{6.14.8}$$

The term enclosed by the last set of parentheses involves the surface curvature and unit normal vector on the right-hand side.

6.14.2 *Surface divergence*

The surface divergence of $\mathbf{T}$, is a vector given by

$$\begin{aligned}\boldsymbol{\psi} = \Big(\frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}\, T^{mj} + \Gamma^j_{mk}\, T^{im}\Big)\, \mathrm{trace}(\mathbf{g}^k \otimes \mathbf{g}_i)\, \mathbf{g}_j \\ -T^{ij}\,\big(\, B_{ik}\, \mathrm{trace}(\mathbf{g}^k \otimes \mathbf{n})\, \mathbf{g}_j + B_{jk}\, \mathrm{trace}(\mathbf{g}^k \otimes \mathbf{g}_i)\, \mathbf{n}\big).\end{aligned} \tag{6.14.9}$$

Setting

$$\mathrm{trace}(\mathbf{g}^k \otimes \mathbf{g}_i) = \delta_{ki}, \qquad \mathrm{trace}(\mathbf{g}^k \otimes \mathbf{n}) = 0, \tag{6.14.10}$$

we recover expression (6.13.11).

Exercise

6.14.1 Derive an expression for the surface gradient over a sphere.

6.15 *Surface divergence theorem*

Consider a surface patch, $\mathcal{P}$, enclosed by a closed loop, $\mathcal{C}$, and a vector function of position defined over the surface, $\mathbf{u}$, as shown in Figure

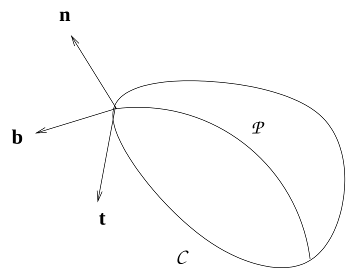

FIGURE 6.15.1 Illustration of a surface patch, denoted by $\mathcal{P}$, enclosed by a closed loop, denoted by $\mathcal{C}$. A surface divergence theorem can be established.

6.15.1. The Gauss divergence theorem for a surface provides us with the identity

$$\iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot (\mathbf{P} \cdot \mathbf{u}) \, \mathrm{d}S = \int_{\mathcal{C}} \mathbf{b} \cdot \mathbf{u} \, \mathrm{d}\ell, \tag{6.15.1}$$

where $\widehat{\boldsymbol{\nabla}} = \mathbf{P} \cdot \boldsymbol{\nabla}$ is the tangential gradient operator, $\mathrm{d}S$ is a differential surface area, ℓ is the arc length measured around $\mathcal{C}$, $\mathbf{b}$ is a tangential unit vector,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{6.15.2}$$

$\mathbf{n}$ is the unit vector normal to the surface, and $\mathbf{t}$ is the unit vector that is tangential to the surface and also tangential to $\mathcal{C}$, as shown in Figure 6.15.1. The integrand on the left-hand side of (6.15.1) is the surface divergence of the tangential part of $\mathbf{u}$.

A proof of the theorem will be presented later in this section in terms of Stokes's circulation theorem.

6.15.1 Surface coordinates

In surface curvilinear coordinates, the surface divergence theorem reads

$$\iint_{\mathcal{P}} u^i_{,i} \, \mathrm{d}S = \int_{\mathcal{C}} b_i \, u^i \, \mathrm{d}\ell, \tag{6.15.3}$$

where summation is implied over the repeated index, i. The integrand on the left-hand side is the ith surface covariant derivative of the ith contravariant components of $\mathbf{u}$, given by

$$u^i_{,i} = \frac{\partial u^i}{\partial x^i} + \Gamma^i_{ik}\, u^k, \qquad (6.15.4)$$

where summation is implied over the repeated indices, i and k.

6.15.2 Surface tensor field

In the case of a surface tensor field, $\mathbf{T}$, the Gauss divergence theorem provides us with the identity

$$\iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot (\mathbf{P} \cdot \mathbf{T})\, \mathrm{d}S = \int_{\mathcal{C}} \mathbf{b} \cdot \mathbf{T}\, \mathrm{d}\ell. \qquad (6.15.5)$$

The integrand on the left-hand side is the surface divergence of $\mathbf{T}$.

Referring to surface curvilinear coordinates, we invoke expression (6.13.15) and obtain

$$\iint_{\mathcal{P}} \big(T^{ij}_{,i}\, \mathbf{g}_j - T^{ij}\, B_{ji}\, \mathbf{n}\big)\, \mathrm{d}S = \int_{\mathcal{C}} b_j T^{ji}\, \mathrm{d}\ell. \qquad (6.15.6)$$

Note the tangential and normal components inside the integral on the left-hand side.

6.15.3 Proof by Stokes' circulation theorem

Stokes' theorem states that the flow rate of the curl of a differentiable function, $\mathbf{f}$, across $\mathcal{D}$, is equal to the circulation of the function along $\mathcal{C}$,

$$\iint_{\mathcal{D}} \big(\boldsymbol{\nabla} \times \mathbf{f}\big) \cdot \mathbf{n}\, \mathrm{d}S = \oint_{\mathcal{C}} \mathbf{f} \cdot \mathbf{t}\, \mathrm{d}\ell. \qquad (6.15.7)$$

Setting $\mathbf{f} = \mathbf{n} \times \mathbf{u}$, we obtain

$$\iint_{\mathcal{D}} \mathbf{n} \cdot \big(\boldsymbol{\nabla} \times (\mathbf{n} \times \mathbf{u})\big)\, \mathrm{d}S = \oint_{\mathcal{C}} (\mathbf{n} \times \mathbf{u}) \cdot \mathbf{t}\, \mathrm{d}\ell. \qquad (6.15.8)$$

Using the properties of the mixed triple product, we find that the right-hand side of (6.15.8) is equal to the right-hand side of (6.15.1). We will demonstrate that the left-hand sides are also equal, that is,

$$\mathbf{n} \cdot \big(\boldsymbol{\nabla} \times (\mathbf{n} \times \mathbf{u})\big) = \mathbf{P} \cdot \boldsymbol{\nabla} \cdot (\mathbf{P} \cdot \mathbf{u}). \tag{6.15.9}$$

The proof is best carried out working in Cartesian coordinates and index notation.

In Cartesian index notation, the left-hand side of (6.15.9), denoted by $\mathcal{L}$, reads

$$\begin{aligned}\mathcal{L} &= n_i\, \epsilon_{ijk} \frac{\partial(\epsilon_{kpq} n_p u_q)}{\partial x_j} \\ &= n_i\, \epsilon_{ijk}\epsilon_{pqk}\, n_p \frac{\partial u_q}{\partial x_j} + n_i\, \epsilon_{ijk}\epsilon_{pqk}\, u_q \frac{\partial n_p}{\partial x_j}.\end{aligned} \tag{6.15.10}$$

Using the properties of the Levi–Civita symbol, we obtain

$$\mathcal{L} = n_p\, n_p \frac{\partial u_q}{\partial x_q} - n_q\, n_p \frac{\partial u_q}{\partial x_p} + n_p\, u_q \frac{\partial n_p}{\partial x_q} - n_q\, u_q \frac{\partial n_p}{\partial x_p}. \tag{6.15.11}$$

Setting $n_p n_p = 1$ and the third term on the right-hand side to zero, we obtain

$$\mathcal{L} = \frac{\partial u_q}{\partial x_q} - n_q\, n_p \frac{\partial u_q}{\partial x_p} - u_n \frac{\partial n_p}{\partial x_p}, \tag{6.15.12}$$

where $u_n = n_q u_q$ is the normal component of $\mathbf{u}$.

In index notation, the right-hand side of (6.15.9), denoted by $\mathcal{R}$, reads

$$\mathcal{R} = (\delta_{ij} - n_i n_j)\, \frac{\partial}{\partial x_j} \Big((\delta_{iq} - n_i n_q)\, u_q \Big). \tag{6.15.13}$$

Expanding the derivatives, we find that

$$\mathcal{R} = (\delta_{ij} - n_i n_j) \Big(\frac{\partial u_i}{\partial x_j} - n_i n_q \frac{\partial u_q}{\partial x_j} - u_q n_q \frac{\partial n_i}{\partial x_j} - n_i u_q \frac{\partial n_q}{\partial x_j} \Big). \tag{6.15.14}$$

Carrying out the multiplication of the terms inside the two parentheses, we obtain

$$\mathcal{R} = \frac{\partial u_i}{\partial x_i} - n_i n_q \frac{\partial u_q}{\partial x_i} - u_n \frac{\partial n_i}{\partial x_i} \tag{6.15.15}$$
$$-n_i u_q \frac{\partial n_q}{\partial x_i} - n_i n_j \frac{\partial u_i}{\partial x_j} + n_j n_q \frac{\partial u_q}{\partial x_j} + n_i n_j u_N \frac{\partial n_i}{\partial x_j} + n_j u_q \frac{\partial n_q}{\partial x_j}.$$

Only the first three terms on the right-hand side survive due to cancellations and other simplifications, showing that $\mathcal{R} = \mathcal{L}$ and thereby completing the proof.

Exercise

6.15.1 Explain how the surface divergence theorem simplifies in a plane.

6.16 Surface force equilibrium over a membrane

As an application, we consider the equilibrium of forces over a thin membrane and identify the surface tensor $\mathbf{T}$ with the surface tension tensor developing in the membrane due to deformation, denoted by $\boldsymbol{\tau}$.

6.16.1 Surface tension tensor

The tension tensor, $\boldsymbol{\tau}$, is defined such that the in-plane tension (surface traction) exerted on a cross-section of the membrane that is normal to the tangential unit vector $\mathbf{b}$, as shown in Figure 6.5.1, is given by $\mathbf{b} \cdot \boldsymbol{\tau}$. To ensure that the surface traction lies in the tangential plane, we require that

$$\mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{0}, \qquad \boldsymbol{\tau} \cdot \mathbf{n} = \mathbf{0}. \tag{6.16.1}$$

The surface traction, $\mathbf{b} \cdot \boldsymbol{\tau}$, has a normal component, given by $\mathbf{b}\,(\mathbf{b} \cdot \boldsymbol{\tau})$, and a shearing component, given by $(\mathbf{I} - \mathbf{b} \otimes \mathbf{b}) \cdot \boldsymbol{\tau}$.

If a membrane develops isotropic tension, τ, then $\boldsymbol{\tau} = \tau\,\mathbf{P}$, where $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the tangential projection operator and $\mathbf{I}$ is the identity matrix. The shearing component is identically zero.

6.16.2 Surface force balance

A force balance over the surface patch shown in Figure 6.5.1, requires that

$$\int_{\mathcal{C}} \mathbf{b} \cdot \boldsymbol{\tau} \, \mathrm{d}\ell + \iint_{\mathcal{P}} \boldsymbol{\phi} \, \mathrm{d}S = \mathbf{0}, \tag{6.16.2}$$

where $\boldsymbol{\phi}$ is an external surface force density distribution over the membrane. Now using the Gauss divergence theorem stated in (6.15.5), and noting that $\mathbf{P} \cdot \boldsymbol{\tau} = \boldsymbol{\tau}$, we obtain

$$\iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} \, \mathrm{d}S + \iint_{\mathcal{P}} \boldsymbol{\phi} \, \mathrm{d}S = \mathbf{0}, \tag{6.16.3}$$

where $\widehat{\boldsymbol{\nabla}} \equiv \mathbf{P} \cdot \boldsymbol{\nabla}$ is the tangential gradient operator. The integrand on the left-hand side is the surface divergence of the surface tension tensor, $\boldsymbol{\tau}$.

Taking the limit as the patch $\mathcal{P}$ shrinks to a point, or else noting that the shape of $\mathcal{P}$ is arbitrary, we derive a force equilibrium equation

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} + \boldsymbol{\phi} = \mathbf{0}, \tag{6.16.4}$$

expressing a balance between $\boldsymbol{\phi}$ and the surface divergence of $\boldsymbol{\tau}$.

6.16.3 Tangential force balance

Projecting the equilibrium equation (6.16.4) onto the tangential plane, we obtain the tangential equilibrium equation

$$\big(\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} \big) \cdot \mathbf{P} + \boldsymbol{\phi} \cdot \mathbf{P} = \mathbf{0}. \tag{6.16.5}$$

Projecting equation (6.16.4) onto the unit normal vector, we obtain the normal equilibrium equation

$$\big(\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} \big) \cdot \mathbf{n} + \phi_n = 0, \tag{6.16.6}$$

where $\phi_n \equiv \boldsymbol{\phi} \cdot \mathbf{n}$ is the normal component of $\boldsymbol{\phi}$.

6.16.4 Normal force balance

In index notation, the normal component of the equilibrium equation (6.16.4) takes the form

$$\mathcal{P}_{\alpha\beta} \frac{\partial \tau_{\alpha\gamma}}{\partial x_\beta} n_\gamma + \phi_n = 0, \tag{6.16.7}$$

where Greek indices indicate Cartesian coordinates. Rearranging the first term on the left-hand side, we obtain

$$\mathcal{P}_{\alpha\beta} \frac{\partial (\tau_{\alpha\gamma} n_\gamma)}{\partial x_\beta} - \mathcal{P}_{\alpha\beta} \frac{\partial n_\gamma}{\partial x_\beta} \tau_{\alpha\gamma} + \phi_n = 0. \tag{6.16.8}$$

Since $\boldsymbol{\tau} \cdot \mathbf{n} = \mathbf{0}$, the first term on the left-hand side is identically zero, yielding

$$\phi_n = B_{\alpha\gamma}\, \tau_{\alpha\gamma}, \tag{6.16.9}$$

where

$$B_{\alpha\gamma} = P_{\alpha\beta} \frac{\partial n_\gamma}{\partial x_\beta} \tag{6.16.10}$$

are the Cartesian components of the curvature tensor, as shown in (6.4.16). In terms of the double-dot product,

$$\phi_n = \mathbf{B} : \boldsymbol{\tau}. \tag{6.16.11}$$

In the absence of normal load $\phi_n = 0$, the right-hand side must be zero.

6.16.5 Isotropic tension

In the case of an isotropic tension field, $\boldsymbol{\tau} = \tau\, \mathbf{P}$, where τ is the scalar tension. Substituting this expression into (6.16.9), we obtain the simplified balance

$$-\tau\, B_{\alpha\gamma} P_{\alpha\gamma} + \phi_n = 0, \tag{6.16.12}$$

where

$$B_{\alpha\gamma} P_{\alpha\gamma} = \mathrm{trace}(\mathbf{B} \cdot \mathbf{P}) = \mathrm{trace}(\mathbf{B}) = 2\kappa_m \tag{6.16.13}$$

and κ_m is the mean curvature. We have found that

$$\phi_n = 2\kappa_m \tau, \tag{6.16.14}$$

consistent with Laplace's law for the pressure drop across a drop or bubble representing a normal load.

6.16.6 Force equilibrium in surface curvilinear coordinates

Next, we introduce surface curvilinear coordinates and resolve the external force density distribution into tangential and normal components as

$$\boldsymbol{\phi} = \phi^j \mathbf{g}_j + \phi_n \, \mathbf{n}, \tag{6.16.15}$$

where ϕ_n is the normal component. Using expression (6.13.15) for the surface divergence, we derive tangential and normal equilibrium equations in surface curvilinear coordinates,

$$\tau^{ij}_{,i} + \phi^j = 0, \qquad \tau^{ij} B_{ji} - \phi_n = 0, \tag{6.16.16}$$

where a comma indicates the covariant derivative. Comparing the second equation with (6.16.11), we confirm that

$$\mathbf{B} : \boldsymbol{\tau} = \tau^{ij} \mathrm{B}_{ji}. \tag{6.16.17}$$

In the case of an isotropic tension field, we set $\boldsymbol{\tau} = \tau \, \mathbf{P}$ and $\tau^{ij} = \tau \, g^{ij}$, where τ is a constant tension, and find that $g^{ij}_{,i} = 0$ and

$$g^{ij} B_{ji} = 2\kappa_m, \tag{6.16.18}$$

where κ_m is the mean curvature, yielding the normal equilibrium equation shown in (6.16.14).

Exercise

6.16.1 Derive equation (6.16.18).

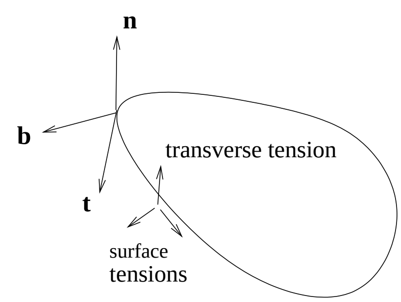

FIGURE 6.17.1 Illustration of a patch of a shell developing tangential tensions, transverse shear tensions, and bending moments.

6.17 Surface force equilibrium over a shell

While tangential surface tensions develop over membranes, as discussed in Section 6.6, transverse shear tensions normal to a material surface and accompanying bending moments develop in addition inside thin shells.

6.17.1 Transverse shear tension

The transverse shear tension can be described by a tangential surface vector field, $\mathbf{q}$, defined such that so that the transverse shear tension in the direction of the unit normal vector, $\mathbf{n}$, exerted on a cross-section of the membrane that is normal to the tangential unit vector $\mathbf{b}$ is given by $\mathbf{b} \cdot \mathbf{q}$, as shown in Figure 6.17.1. By construction,

$$\mathbf{P} \cdot \mathbf{q} = \mathbf{q}. \tag{6.17.1}$$

Extension to three dimensions can be implemented by requiring that $\mathbf{n} \cdot \mathbf{q} = 0$.

The complete interface tension tensor incorporating tangential and transverse shear tensions is given by

$$\boldsymbol{\lambda} \equiv \boldsymbol{\tau} + \mathbf{q} \otimes \mathbf{n}, \tag{6.17.2}$$

where $\boldsymbol{\tau}$ is the tangential tension tensor. Note that

$$\mathbf{P} \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda}, \qquad \mathbf{b} \cdot \boldsymbol{\lambda} = \mathbf{b} \cdot \boldsymbol{\tau} + (\mathbf{b} \cdot \mathbf{q})\,\mathbf{n}, \tag{6.17.3}$$

where the unit vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is defined in Figure 6.17.1.

The vector $\mathbf{q}$ can be resolved into surface contravariant or covariant components as

$$\mathbf{q} = q^i\,\mathbf{g}_i = q_i\,\mathbf{g}^i. \tag{6.17.4}$$

Using these expansions, we write

$$\mathbf{q} \otimes \mathbf{n} = q^i\,\mathbf{g}_i \otimes \mathbf{n} = q_i\,\mathbf{g}^i \otimes \mathbf{n}. \tag{6.17.5}$$

Note that

$$\mathbf{b} \cdot (\mathbf{q} \otimes \mathbf{n}) = q^i\,(\mathbf{b} \cdot \mathbf{g}_i)\,\mathbf{n} = q_i\,(\mathbf{b} \cdot \mathbf{g}^i)\,\mathbf{n}, \tag{6.17.6}$$

representing the transverse shear tension oriented normal to the shell.

6.17.2 Force equilibrium

A force balance over the surface patch shown in Figure 6.17.1 requires that

$$\int_{\mathcal{C}} \mathbf{b} \cdot \boldsymbol{\lambda}\,\mathrm{d}\ell + \iint_{\mathcal{P}} \boldsymbol{\phi}\,\mathrm{d}S = \mathbf{0}, \tag{6.17.7}$$

where $\boldsymbol{\phi}$ is an external force density distribution over the area of the shell, as discussed in Section 6.6 for a membrane.

Now using the Gauss divergence theorem stated in (6.15.5), and recalling that $\mathbf{P} \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda}$, we obtain

$$\iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}\,\mathrm{d}S + \iint_{\mathcal{P}} \boldsymbol{\phi}\,\mathrm{d}S = \mathbf{0}. \tag{6.17.8}$$

The integrand on the left-hand side is the surface divergence of $\boldsymbol{\lambda}$. Taking the limit as the $\mathcal{P}$ shrinks to a point, or else noting that the patch $\mathcal{P}$ is arbitrary, we derive a force equilibrium equation

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda} + \boldsymbol{\phi} = \mathbf{0}, \tag{6.17.9}$$

involving the surface divergence of $\boldsymbol{\lambda}$.

6.17.3 Resolution into tangential and normal components

In index notation, the γ Cartesian component of the surface divergence of the term involving the transverse shear tension is given by

$$[\,\hat{\boldsymbol{\nabla}} \cdot (\mathbf{q} \otimes \mathbf{n})\,]_\gamma = P_{\alpha\beta}\,\frac{\partial(q_\alpha n_\gamma)}{\partial x_\beta}, \tag{6.17.10}$$

where summation is implied over repeated Greek indices denoting Cartesian coordinates; for example, $x_1 = x$, $x_2 = y$, and $x_3 = z$. Expanding the derivative, we obtain

$$[\,\hat{\boldsymbol{\nabla}} \cdot (\mathbf{q} \otimes \mathbf{n})\,]_\gamma = q_\alpha\,P_{\alpha\beta}\,\frac{\partial n_\gamma}{\partial x_\beta} + n_\gamma\,P_{\alpha\beta}\,\frac{\partial q_\alpha}{\partial x_\beta}. \tag{6.17.11}$$

Invoking the definition of the curvature tensor stated in (6.4.15) as $\mathbf{B} = \hat{\boldsymbol{\nabla}}\,\mathbf{n}$, we obtain

$$[\,\hat{\boldsymbol{\nabla}} \cdot (\mathbf{q} \otimes \mathbf{n})\,]_\gamma = q_\alpha\,B_{\alpha\gamma} + n_\gamma\,P_{\alpha\beta}\,\frac{\partial q_\alpha}{\partial x_\beta}. \tag{6.17.12}$$

In vector notation,

$$\hat{\boldsymbol{\nabla}} \cdot (\mathbf{q} \otimes \mathbf{n}) = \mathbf{q} \cdot \mathbf{B} + (\hat{\boldsymbol{\nabla}} \cdot \mathbf{q})\,\mathbf{n}. \tag{6.17.13}$$

The force equilibrium equation (6.17.9) then becomes

$$\hat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} + \mathbf{q} \cdot \mathbf{B} + (\hat{\boldsymbol{\nabla}} \cdot \mathbf{q})\,\mathbf{n} + \boldsymbol{\phi} = \mathbf{0}. \tag{6.17.14}$$

This vector equation can be resolved into its tangential and normal components as

$$(\hat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau}) \cdot \mathbf{P} + \mathbf{q} \cdot \mathbf{B} + \boldsymbol{\phi} \cdot \mathbf{P} = \mathbf{0} \tag{6.17.15}$$

and

$$(\hat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau}) \cdot \mathbf{n} + \hat{\boldsymbol{\nabla}} \cdot \mathbf{q} + \boldsymbol{\phi} \cdot \mathbf{n} = 0. \tag{6.17.16}$$

6.17.4 Bending moments

Bending moments develop inside the cross-section of a thin shell due to in-plane stress distributions. The tangential tensions are the associated stress resultants. The bending moments can be described in terms of a surface Cartesian tensor, $\mathbf{m}$, which is defined so that the bending moment vector exerted on a membrane cross-section that is normal to the tangential unit vector $\mathbf{b}$ is given by

$$\mathbf{n} \times (\mathbf{b} \cdot \mathbf{m}). \tag{6.17.17}$$

To ensure that the bending moments lie in a tangential plane, we require that $\mathbf{n} \cdot \mathbf{m} = \mathbf{0}$ and $\mathbf{m} \cdot \mathbf{n} = \mathbf{0}$.

6.17.5 Torque balance

Performing a torque balance over the surface patch shown in Figure 6.17.1 with respect to an arbitrary reference point, $\mathbf{x}_0$, we write

$$\int_{\mathcal{C}} \mathbf{n} \times (\mathbf{b} \cdot \mathbf{m}) \, \mathrm{d}\ell + \int_{\mathcal{C}} (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{b} \cdot \boldsymbol{\lambda}) \, \mathrm{d}\ell + \iint_{\mathcal{P}} (\mathbf{x} - \mathbf{x}_0) \times \boldsymbol{\phi} \, \mathrm{d}S = \mathbf{0}. \tag{6.17.18}$$

Combining this equation with the equilibrium equation (6.17.9) to eliminate the last integral involving the external force load density function, $\boldsymbol{\phi}$, we obtain the simplified equation

$$\int_{\mathcal{C}} \mathbf{n} \times (\mathbf{b} \cdot \mathbf{m}) \, \mathrm{d}\ell + \int_{\mathcal{C}} (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{b} \cdot \boldsymbol{\lambda}) \, \mathrm{d}\ell - \iint_{\mathcal{P}} (\mathbf{x} - \mathbf{x}_0) \times (\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}) \, \mathrm{d}S = \mathbf{0}. \tag{6.17.19}$$

Next, we use the surface divergence theorem to eliminate terms involving the arbitrary point $\mathbf{x}_0$ due to mutual cancellation, and obtain the simplified form

$$\int_{\mathcal{C}} \mathbf{n} \times (\mathbf{b} \cdot \mathbf{m}) \, \mathrm{d}\ell + \int_{\mathcal{C}} \mathbf{x} \times (\mathbf{b} \cdot \boldsymbol{\lambda}) \, \mathrm{d}\ell - \iint_{\mathcal{P}} \mathbf{x} \times (\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}) \, \mathrm{d}S = \mathbf{0}. \tag{6.17.20}$$

In index notation, the ith Cartesian component of this equation reads

$$\int_{\mathcal{C}} b_p \, (m_{pk} \epsilon_{ijk} \, n_j \,) \, \mathrm{d}\ell + \int_{\mathcal{C}} b_p (\lambda_{pk} \epsilon_{ijk} \, x_j) \, \mathrm{d}\ell - \iint_{\mathcal{P}} \epsilon_{ijk} x_j \, [\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}]_k \, \mathrm{d}S = 0. \tag{6.17.21}$$

Reverting to vector notation, we obtain

$$\int_{\mathcal{C}} \mathbf{b} \cdot (\mathbf{m} \times \mathbf{n})\, \mathrm{d}\ell + \int_{\mathcal{C}} \mathbf{b} \cdot (\boldsymbol{\lambda} \times \mathbf{x})\, \mathrm{d}\ell - \iint_{\mathcal{P}} (\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}) \times \mathbf{x}\, \mathrm{d}S = \mathbf{0}. \tag{6.17.22}$$

Now applying the surface divergence theorem to the first two integrals, we obtain

$$\iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot (\mathbf{m} \times \mathbf{n})\, \mathrm{d}S + \iint_{\mathcal{P}} \widehat{\boldsymbol{\nabla}} \cdot (\boldsymbol{\lambda} \times \mathbf{x}) - \iint_{\mathcal{P}} (\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}) \times \mathbf{x}\, \mathrm{d}S = \mathbf{0}. \tag{6.17.23}$$

All three integrals in this equation are performed over the patch surface area.

6.17.6 Differential torque balance

Discarding the integral signs in the integral balance (6.17.23) and rearranging, we obtain the differential torque balance

$$\widehat{\boldsymbol{\nabla}} \cdot (\mathbf{m} \times \mathbf{n}) = -\widehat{\boldsymbol{\nabla}} \cdot (\boldsymbol{\lambda} \times \mathbf{x}) + (\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\lambda}) \times \mathbf{x}. \tag{6.17.24}$$

In index notation, the ith Cartesian component of this equation reads

$$\widehat{\nabla}_j (m_{jp} \epsilon_{ipk} n_k) = -\widehat{\nabla}_j (\lambda_{jp} \epsilon_{ipk} x_k) + (\widehat{\nabla}_j \lambda_{jp})\, \epsilon_{ipk} x_k. \tag{6.17.25}$$

Simplifying the right-hand side, we obtain

$$\widehat{\nabla}_j (m_{jp} \epsilon_{ipk} n_k) = -\epsilon_{ipk} \lambda_{jp}\, \widehat{\nabla}_j x_k. \tag{6.17.26}$$

Reverting to vector–matrix notation, we obtain the equilibrium condition

$$\widehat{\boldsymbol{\nabla}} \cdot (\mathbf{m} \times \mathbf{n}) = -(\boldsymbol{\lambda}^{\mathrm{T}} \cdot \widehat{\boldsymbol{\nabla}}) \times \mathbf{x}, \tag{6.17.27}$$

where the superscript T denotes the matrix transpose.

6.17.7 Resolution into tangential and normal components

Expanding the derivative on the left-hand side of (6.17.26), we obtain

$$\epsilon_{ipk}\big(n_k\widehat{\nabla}_j m_{jp} + m_{jp}\widehat{\nabla}_j n_k\big) = \epsilon_{ipk}\big(n_k\widehat{\nabla}_j m_{jp} + m_{jp}B_{jk}\big), \quad (6.17.28)$$

where B_{jk} is a Cartesian component of the curvature tensor.

Since $\partial x_k/\partial x_l = \delta_{kl}$, the right-hand side of (6.17.26) can be manipulated as follows:

$$a_i \equiv -\epsilon_{ipk}\,\lambda_{jp}\,P_{jl}\frac{\partial x_k}{\partial x_l} = -\epsilon_{ipk}\,\lambda_{jp}\,P_{jk}. \quad (6.17.29)$$

Resolving $\boldsymbol{\lambda}$ into two constituents, we obtain

$$a_i = -\epsilon_{ipk}\,\tau_{jp}\,P_{jk} - \epsilon_{ipk}\,q_j n_p\,P_{jk} = -\epsilon_{ipk}\,\tau_{kp} - \epsilon_{ipk}\,q_k n_p. \quad (6.17.30)$$

The last term involves the unit normal vector.

Substituting (6.17.28) and (6.17.30) into (6.17.26), we obtain

$$\epsilon_{ipk}\big(n_k\widehat{\nabla}_j m_{jp} + m_{jp}\,B_{jk}\big) = -\epsilon_{ipk}\,\tau_{kp} - \epsilon_{ipk}\,q_k n_p. \quad (6.17.31)$$

Rearranging, we obtain

$$\epsilon_{ipk}\big(n_k\widehat{\nabla}_j m_{jp} + q_k\,n_p\big) = -\epsilon_{ipk}\,(\tau_{kp} + B_{kj}m_{jp}). \quad (6.17.32)$$

This equation is satisfied when the tensors inside the parentheses on the left- and right-hand sides are symmetric with respect to the indices p and k, which is true when

$$q_k = \widehat{\nabla}_j m_{jk} \quad (6.17.33)$$

and

$$\tau_{kp} + B_{kj}m_{jp} = \tau_{pk} + B_{pj}m_{jk}. \quad (6.17.34)$$

With regard to (6.17.33), we ensure that $\mathbf{q}\cdot\mathbf{n} = 0$ by setting

$$\mathbf{q} = (\,\widehat{\boldsymbol{\nabla}}\cdot\mathbf{m}\,)\cdot\mathbf{P}. \quad (6.17.35)$$

With regard to (6.17.34), we derive an expression for the antisymmetric part of the in-plane tension tensor,

$$\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathrm{T}} = -\mathbf{B} \cdot \mathbf{m} + \mathbf{m}^{\mathrm{T}} \cdot \mathbf{B} = -\mathbf{B} \cdot \mathbf{m} + (\mathbf{B} \cdot \mathbf{m})^{\mathrm{T}}, \quad (6.17.36)$$

where the superscript T denotes the matrix transpose. The right-hand side involves the antisymmetric part of $\mathbf{B} \cdot \mathbf{m}$. When $\mathbf{m}$ is symmetric, the tensor $\boldsymbol{\tau}$ is also symmetric.

Exercises

6.17.1 Introduce a surface tensor field with components $M_{ij} = \epsilon_{ikp} n_k m_{jp}$ and show that $\mathbf{m} \times \mathbf{n} = -\mathbf{M}^{\mathrm{T}}$ and the equilibrium condition (6.17.27) becomes $\hat{\boldsymbol{\nabla}} \cdot \mathbf{M}^{\mathrm{T}} = (\boldsymbol{\lambda}^{\mathrm{T}} \cdot \hat{\boldsymbol{\nabla}}) \times \mathbf{x}$.

6.17.2 Derive equation (6.17.27) from (6.17.26).

6.18 Axisymmetric shells

Consider an axisymmetric shell whose shape is generated by rotating a curve around the x axis, as shown in Figure 6.18.1. To describe the profile of the shell, we introduce cylindrical polar coordinates consisting of the axial position, x, the distance from the x axis, σ, and the azimuthal angle, φ, measured around the x axis with origin in the xy plane.

6.18.1 Arc length parametrization

The axisymmetric shape can be parametrized in terms of the arc length measured along the contour of the membrane in an azimuthal plane, ℓ, as shown in Figure 6.18.1.

The unit vector that is tangential to the membrane and lies in an azimuthal plane defined by a certain value of the azimuthal angle, φ, is denoted by $\mathbf{t}_\ell$, whereas the azimuthal unit vector is denoted by $\mathbf{t}_\varphi$. The unit vector normal to the membrane, $\mathbf{n}$, points outward, as shown in Figure 6.18.1. The triplet of vectors, $\mathbf{t}_\ell$, $\mathbf{t}_\varphi$, and $\mathbf{n}$ define local orthogonal coordinates.

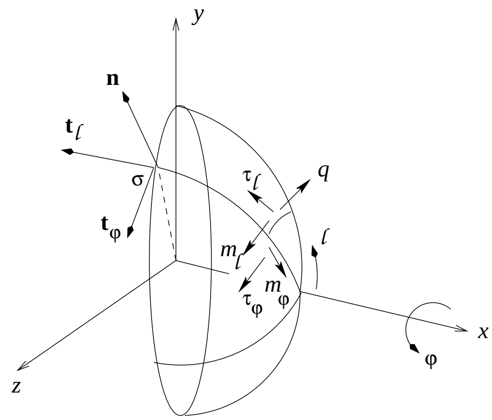

FIGURE 6.18.1 Illustration of an axisymmetric shell enclosed by a membrane.

The axial position of the shell in an azimuthal plane can described by a function

$$x = \xi(\ell), \tag{6.18.1}$$

and the radial position can described by either one of the functions

$$\sigma = \varsigma(\ell) = \Sigma(x). \tag{6.18.2}$$

We find that

$$\mathbf{t}_\ell = \begin{bmatrix} \xi' \\ \varsigma' \cos\varphi \\ \varsigma' \sin\varphi \end{bmatrix}, \quad \mathbf{t}_\varphi = \begin{bmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} \varsigma' \\ -\xi' \cos\varphi \\ -\xi' \sin\varphi \end{bmatrix}, \tag{6.18.3}$$

where $\xi' \equiv \mathrm{d}\xi/\mathrm{d}\ell$, $\varsigma' \equiv \mathrm{d}\varsigma/\mathrm{d}\ell$, and $\xi'^2 + \varsigma'^2 = 1$.

6.18.2 Principal curvatures

The principal curvatures of the interface in an azimuthal and its conjugate plane are denoted by κ_ℓ and κ_φ. The curvature tensor is given by

$$\mathbf{B} = \kappa_\ell \, \mathbf{t}_\ell \otimes \mathbf{t}_\ell + \kappa_\varphi \, \mathbf{t}_\varphi \otimes \mathbf{t}_\varphi. \tag{6.18.4}$$

Using fundamental relations of differential geometry, we find that

$$\kappa_\ell = -\frac{\pm\,\varsigma''}{\sqrt{1-\varsigma'^2}} = -\frac{\pm\,\Sigma''}{(1+\Sigma'^2)^{3/2}} \tag{6.18.5}$$

and

$$\kappa_\varphi = -\frac{1}{\sigma}\,\frac{\mathrm{d}x}{\mathrm{d}\ell} = \pm\frac{1}{\sigma}\,\sqrt{1-\varsigma'^2} = \pm\frac{1}{\sigma}\,\frac{1}{\sqrt{1+\Sigma'^2}}, \tag{6.18.6}$$

where $\Sigma' \equiv \mathrm{d}\Sigma/\mathrm{d}x$. The plus sign of $\pm$ is selected when $\mathrm{d}x/\mathrm{d}\ell < 0$, and the minus sign otherwise.

6.18.3 Codazzi's equation

Expressions (6.18.5) and (6.18.6) are consistent with Codazzi's equation

$$\kappa_\ell = \frac{\mathrm{d}(\sigma\,\kappa_\varphi)}{\mathrm{d}\sigma}, \tag{6.18.7}$$

which allows us to compute one of the principal curvatures in terms of the other. Rearranging (6.18.7), we obtain

$$\frac{\mathrm{d}\kappa_\varphi}{\mathrm{d}\sigma} = \frac{\kappa_\ell - \kappa_\varphi}{\sigma}. \tag{6.18.8}$$

Applying (6.18.8) at the axis of symmetry where $\sigma = 0$, and using the rule de l'Hôspital to evaluate the right-hand side, we obtain

$$2\left(\frac{\mathrm{d}\kappa_\varphi}{\mathrm{d}\ell}\right)_{\sigma=0} = \left(\frac{\mathrm{d}\kappa_\ell}{\mathrm{d}\ell}\right)_{\sigma=0}. \tag{6.18.9}$$

Differentiating (6.18.8) with respect to ℓ and working in a similar fashion, we find that

$$3\left(\frac{\mathrm{d}^2\kappa_\varphi}{\mathrm{d}\ell^2}\right)_{\sigma=0} = \left(\frac{\mathrm{d}^2\kappa_\ell}{\mathrm{d}\ell^2}\right)_{\sigma=0}. \tag{6.18.10}$$

6.18.4 Tensions and bending moments

Working under the auspices of thin-shell theory, we consider the mid-surface of the membrane and introduce (a) the azimuthal and meridional tensions, τ_ℓ and τ_φ, which are the principal tensions of the in-plane stress resultants, (b) the transverse shearing tension, q, exerted on a cross-section of the membrane that is normal to the x axis, and (c) the meridional and azimuthal bending moments, m_ℓ and m_φ, as depicted in Figure 6.18.1.

6.18.5 Shearing-tension surface vector field

The tangential surface vector field $\mathbf{q}$ describing the shear tension, introduced in Section 6.17, is given by

$$\mathbf{q} = q\,\mathbf{t}_\ell. \tag{6.18.11}$$

We find that

$$\mathbf{q}\cdot\mathbf{B} = q\,\kappa_\ell\,\mathbf{t}_\ell. \tag{6.18.12}$$

The surface divergence of $\mathbf{q}$ is given by

$$\widehat{\boldsymbol{\nabla}}\cdot\mathbf{q} = \mathbf{t}_\ell\cdot\frac{\partial(q\,\mathbf{t}_\ell)}{\partial\ell} + \frac{1}{\sigma}\,\mathbf{t}_\varphi\cdot\frac{\partial(q\mathbf{t}_\ell)}{\partial\varphi}. \tag{6.18.13}$$

Expanding the derivatives and simplifying, we obtain

$$\widehat{\boldsymbol{\nabla}}\cdot\mathbf{q} = \frac{\partial q}{\partial\ell} + q\,\frac{1}{\sigma}\,\mathbf{t}_\varphi\cdot\frac{\partial\mathbf{t}_\ell}{\partial\varphi}. \tag{6.18.14}$$

With regard to the last term on the right-hand side, we use expressions (6.18.3) to compute

$$\mathbf{t}_\varphi\cdot\frac{\partial\mathbf{t}_\ell}{\partial\varphi} = \frac{\partial\sigma}{\partial\ell} = \varsigma', \tag{6.18.15}$$

and obtain

$$\widehat{\boldsymbol{\nabla}}\cdot\mathbf{q} = \frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial\ell}. \tag{6.18.16}$$

Later in this section, this expression will be substituted into the interfacial force balance.

6.18.6 In-plane tension tensor

The in-plane tension tensor is given by

$$\boldsymbol{\tau} = \tau_\ell \, \mathbf{t}_\ell \otimes \mathbf{t}_\ell + \tau_\varphi \, \mathbf{t}_\varphi \otimes \mathbf{t}_\varphi, \tag{6.18.17}$$

where τ_ℓ and τ_φ depend on ℓ but not on φ. The surface divergence is given by

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} = \mathbf{t}_\ell \cdot \frac{\partial \boldsymbol{\tau}}{\partial \ell} + \frac{1}{\sigma} \, \mathbf{t}_\varphi \cdot \frac{\partial \boldsymbol{\tau}}{\partial \varphi}. \tag{6.18.18}$$

Substituting expression (6.18.17) for $\boldsymbol{\tau}$, expanding the derivatives, and neglecting terms that are identically zero, we find that

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} = \frac{\mathrm{d}\tau_\ell}{\mathrm{d}\ell} \, \mathbf{t}_\ell + \tau_\ell \, \frac{\partial \mathbf{t}_\ell}{\partial \ell} + \frac{1}{\sigma} \left(\tau_\ell \, \mathbf{t}_\varphi \cdot \frac{\partial \mathbf{t}_\ell}{\partial \varphi} \, \mathbf{t}_\ell + \tau_\varphi \, \frac{\partial \mathbf{t}_\varphi}{\partial \varphi} \right). \tag{6.18.19}$$

With regard to the last term on the right-hand side, we use expressions (6.18.3) and (6.18.6) to compute

$$\frac{\partial \mathbf{t}_\varphi}{\partial \varphi} = - \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix} = -\varsigma' \, \mathbf{t}_\ell + \xi' \, \mathbf{n} = -\varsigma' \, \mathbf{t}_\ell - \sigma \, \kappa_\varphi \, \mathbf{n}. \tag{6.18.20}$$

Substituting this expression into (6.18.19), setting $\partial \mathbf{t}_\ell / \partial \ell = -\kappa_\ell \, \mathbf{n}$, and recalling (6.18.15), we obtain

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} = \frac{\mathrm{d}\tau_\ell}{\mathrm{d}\ell} \, \mathbf{t}_\ell - \tau_\ell \, \kappa_\ell \, \mathbf{n} + \frac{1}{\sigma} \left(\tau_\ell \, \varsigma' \, \mathbf{t}_\ell - \tau_\varphi (\varsigma' \, \mathbf{t}_\ell + \sigma \, \kappa_\varphi \, \mathbf{n}) \right). \tag{6.18.21}$$

Separating the normal from the tangential components, we obtain

$$\widehat{\boldsymbol{\nabla}} \cdot \boldsymbol{\tau} = -(\kappa_\ell \, \tau_\ell + \kappa_\varphi \, \tau_\varphi) \, \mathbf{n} + \left(\frac{\mathrm{d}\tau_\ell}{\mathrm{d}\ell} + \frac{1}{\sigma} \, \frac{\mathrm{d}\sigma}{\mathrm{d}\ell} \, (\tau_\ell - \tau_\varphi) \right) \mathbf{t}_\ell. \tag{6.18.22}$$

Substituting expressions (6.18.12), (6.18.16), and (6.18.22) into the interfacial force balance (6.17.14), we obtain

$$-\big(\,\kappa_\ell\,\tau_\ell+\kappa_\varphi\,\tau_\varphi-\frac{1}{\sigma}\,\frac{\mathrm{d}(\sigma q)}{\mathrm{d}\ell}\,\big)\,\mathbf{n} +\big(\frac{\mathrm{d}\tau_\ell}{\mathrm{d}\ell}+\frac{1}{\sigma}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\ell}\,(\tau_\ell-\tau_\varphi)+q\,\kappa_\ell\,\big)\,\mathbf{t}_\ell+\boldsymbol{\phi}=\mathbf{0}. \quad (6.18.23)$$

Now resolving the distributed load into normal and tangential components,

$$\boldsymbol{\phi}=\phi_n\,\mathbf{n}+\phi_\ell\,\mathbf{t}_\ell, \quad (6.18.24)$$

we obtain the corresponding equilibrium equations

$$\phi_n=\kappa_\ell\,\tau_\ell+\kappa_\varphi\,\tau_\varphi-\frac{1}{\sigma}\,\frac{\mathrm{d}(\sigma q)}{\mathrm{d}\ell} \quad (6.18.25)$$

and

$$\phi_\ell=-\frac{\mathrm{d}\tau_\ell}{\mathrm{d}\ell}-\frac{1}{\sigma}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\ell}\,(\tau_\ell-\tau_\varphi)-\kappa_\ell\,q=-\frac{1}{\sigma}\,\frac{\mathrm{d}(\sigma\,\tau_\ell)}{\mathrm{d}\ell}+\frac{\tau_\varphi}{\sigma}\,\frac{\mathrm{d}\sigma}{d\ell}-\kappa_\ell\,q. \quad (6.18.26)$$

6.18.7 Torque equilibrium

The tensor of bending moments is given by

$$\mathbf{m}=m_\ell\,\mathbf{t}_\ell\otimes\mathbf{t}_\ell+m_\varphi\,\mathbf{t}_\varphi\otimes\mathbf{t}_\varphi. \quad (6.18.27)$$

The counterpart of equation (6.18.22) for the surface divergence is

$$\widehat{\boldsymbol{\nabla}}\cdot\mathbf{m}=-(\kappa_\ell\,m_\ell+\kappa_\varphi\,m_\varphi)\,\mathbf{n}+\big(\,\frac{\mathrm{d}m_\ell}{\mathrm{d}\ell}+\frac{1}{\sigma}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\ell}\,(m_\ell-m_\varphi)\,\big)\,\mathbf{t}_\ell. \quad (6.18.28)$$

Substituting this expression into (6.17.35), we obtain

$$q=\frac{1}{\sigma}\,\Big(\frac{\mathrm{d}(\sigma\,m_\ell)}{\mathrm{d}\ell}-m_\varphi\,\frac{\mathrm{d}\sigma}{\mathrm{d}\ell}\Big)=\frac{1}{\sigma}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\ell}\,\Big(\frac{\mathrm{d}(\sigma\,m_\ell)}{d\sigma}-m_\varphi\Big). \quad (6.18.29)$$

Constitutive equations for the elastic tensions and bending moments are required for closure.

Exercises

6.18.1 Derive expressions (6.18.9) and (6.18.10).

6.18.2 Derive equations (6.18.25), (6.18.26), and (6.18.29) by performing force and torque balances over a small section of the membrane that is confined between (a) two adjacent azimuthal planes passing through the x axis, and (b) two parallel planes that are perpendicular to the x axis and enclose a small section of the membrane in a azimuthal plane, as shown in Figure 6.18.1.

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Tensors Unravelled

C. Pozrikidis

In this concise yet comprehensive book, the author introduces the notion of tensors with reference to arbitrary bases in Cartesian or non-Cartesian, rectilinear or curvilinear coordinates. The description of tensors in terms of their components in a specified base is emphasized and transformation rules are established. Noteworthy features include the following:

- An introduction to tensors is presented preceding the discussion of curvilinear coordinates
- The concept of uniadic, dyadic, and multiadic bases is emphasized with reference to one-, two-, and higher-index tensor arrays
- Non-Cartesian coordinates are discussed from the viewpoint of applied mathematics and engineering, and comprehensive expressions from differential calculus are derived
- Convected coordinates are discussed and expressions for Green's function of the convection–diffusion equation are derived
- The apparatus of surface curvilinear coordinates is discussed in terms of the curvature tensor with applications to the mechanics of membranes and shells
- Theory and computation are discussed alongside by way of computer codes that illustrate and implements theoretical predictions
- A suite of computer codes that confirm theoretical derivations and encode methods for computing solutions of selected differential equations accompany the text

Chester & Bennington