

rotates with the fluid, we find that the Coriolis force is zero, and the centrifugal force is $-\rho u_\phi^2/\sigma$. The temptation to interpret the centripetal acceleration force, in the inertial system as the centrifugal force is then apparent.

Problems

3.2.1 Angular momentum balance with respect to an arbitrary point in space

Write the counterpart of the balance (3.1.24) when the angular momentum and torque are computed with respect to an arbitrary point, \mathbf{x}_0 . Then proceed as in the text to derive (3.2.13).

3.2.2 Hydrodynamic volume force in polar coordinates

Derive the components of the hydrodynamic volume force, $\Sigma \equiv \nabla \cdot \boldsymbol{\sigma}$, in (a) cylindrical polar and (b) spherical polar coordinates.

3.3 Energy balances

A differential energy balance can be obtained by projecting Cauchy's equation of motion onto the velocity vector at an arbitrary point in a flow. Projecting the Eulerian form,

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}, \quad (3.3.1)$$

and rearranging the left-hand side, we obtain

$$\frac{1}{2} \rho \left(\frac{\partial |\mathbf{u}|^2}{\partial t} + \mathbf{u} \cdot \nabla |\mathbf{u}|^2 \right) = \mathbf{u} \cdot (\nabla \cdot \boldsymbol{\sigma}) + \rho \mathbf{g} \cdot \mathbf{u}, \quad (3.3.2)$$

where $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ is the square of the magnitude of the velocity. Combining this equation with the continuity equation (3.2.4),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3.3.3)$$

to modify the left-hand side, and rearranging the right-hand side, we obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{u}|^2 \right) + \nabla \cdot \left(\frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} \right) = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) - \boldsymbol{\sigma} : \nabla \mathbf{u} + \rho \mathbf{g} \cdot \mathbf{u} \quad (3.3.4)$$

for a compressible or incompressible fluid. The double-dot product of two matrices, in this case $\boldsymbol{\sigma}$ and $\nabla \mathbf{u}$, is a scalar defined in Section A.4, Appendix A,

$$\boldsymbol{\sigma} : \nabla \mathbf{u} \equiv \sigma_{ij} \frac{\partial u_j}{\partial x_i}, \quad (3.3.5)$$

where summation is implied over the repeated indices i and j . The balance (3.3.4) applies for symmetric and unsymmetric stress tensors.

Integral energy balance

An integral or macroscopic energy balance arises by integrating the differential energy balance over a fixed control volume, \mathcal{V}_c , bounded by a surface or a collection of surfaces, denoted by \mathcal{D} , as shown in Figure 3.2.1. Integrating (3.3.4) and using the divergence theorem, we obtain

$$\begin{aligned} \iiint_{\mathcal{V}_c} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{u}|^2 \right) dV &= \iint_{\mathcal{D}} \left(\frac{1}{2} \rho |\mathbf{u}|^2 \right) \mathbf{u} \cdot \mathbf{n}^{\text{in}} dS \\ &\quad - \iint_{\mathcal{D}} \mathbf{f}^{\text{in}} \cdot \mathbf{u} dS - \iiint_{\mathcal{V}_c} \boldsymbol{\sigma} : \nabla \mathbf{u} dV + \iiint_{\mathcal{V}_c} \rho \mathbf{g} \cdot \mathbf{u} dV, \end{aligned} \quad (3.3.6)$$

where $\mathbf{f}^{\text{in}} = \mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma}$ is the traction and \mathbf{n}^{in} is the normal unit vector pointing *into* the control volume. The four terms on the right-hand side of (3.3.6) represent, respectively,

1. The rate of supply of kinetic energy *into* the control volume by convection,
2. The rate of working of the traction at the boundary of the control volume.
3. The rate of volumetric energy dissipation.
4. The rate of working against the body force.

We have found that the rate of volumetric energy dissipation due to the fluid motion is given by the double-dot product of the stress tensor and the velocity gradient tensor, $\boldsymbol{\sigma} : \nabla \mathbf{u}$. Constitutive equations for the stress tensor must be introduced to evaluate this term, as discussed in Section 3.5.

Rate of working against the body force

When the fluid is incompressible, $D\rho/Dt = 0$, and the fluid density is uniform throughout the domain of flow, the last volume integral in (3.3.6), expressing the rate of working against the body force, can be transformed into a surface integral over the boundary, \mathcal{D} , by writing

$$\mathcal{W}_B \equiv \iiint_{\mathcal{V}_c} \rho \mathbf{g} \cdot \mathbf{u} dV = \rho \iiint_{\mathcal{V}_c} \nabla \cdot [(\mathbf{g} \cdot \mathbf{x}) \mathbf{u}] dV = -\rho \iint_{\mathcal{D}} (\mathbf{g} \cdot \mathbf{x})(\mathbf{u} \cdot \mathbf{n}^{\text{in}}) dS, \quad (3.3.7)$$

where \mathbf{n}^{in} is the normal unit vector pointing into the control volume.

If the normal component of the velocity obeys the no-penetration boundary condition for a translating body, $\mathbf{u} \cdot \mathbf{n}^{\text{in}} = \mathbf{V} \cdot \mathbf{n}^{\text{in}}$, we may apply the divergence theorem to find that

$$\mathcal{W}_B = \rho V_{\text{cv}} \mathbf{g} \cdot \mathbf{V}, \quad (3.3.8)$$

where V_{cv} is the volume of the control volume and \mathbf{V} is a constant velocity. The last term in (3.3.6) may then be identified with the rate of working necessary to elevate the fluid inside the control volume with velocity \mathbf{V} .

3.3.1 Energy generation inside a fluid parcel

The total energy of the fluid residing inside a parcel is comprised of (a) the kinetic energy due to the motion of the fluid, (b) the potential energy due to an external body force field, and (c) the internal thermodynamic energy. The instantaneous kinetic and potential energies are given by

$$\mathcal{E}_k \equiv \frac{1}{2} \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{u} \, dV, \quad \mathcal{E}_p \equiv - \iiint_{\text{parcel}} \rho \mathbf{X} \cdot \mathbf{g} \, dV, \quad (3.3.9)$$

where \mathbf{X} is the position of a point particle occupying the parcel. The negative sign before the integral of the potential energy takes into consideration that potential energy is gained when a body or particle moves against the body force.

Taking the derivative of the kinetic energy integral, transferring the derivative inside the integral as a material derivative, and recalling that, because of mass conservation, $D(\rho \, dV)/Dt = 0$, we express the rate of change of the kinetic energy in terms of the point-particle acceleration,

$$\frac{d\mathcal{E}_k}{dt} = \frac{1}{2} \iiint_{\text{parcel}} \rho \frac{D(\mathbf{u} \cdot \mathbf{u})}{Dt} \, dV = \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} \, dV. \quad (3.3.10)$$

Now we use the equation of motion to express the acceleration in terms of the stress tensor and body force, obtaining

$$\frac{d\mathcal{E}_k}{dt} = \iiint_{\text{parcel}} \mathbf{u} \cdot (\nabla \cdot \boldsymbol{\sigma}) \, dV + \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} \, dV. \quad (3.3.11)$$

Further manipulation yields

$$\frac{d\mathcal{E}_k}{dt} = \iiint_{\text{parcel}} \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) \, dV - \iiint_{\text{parcel}} \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV + \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} \, dV. \quad (3.3.12)$$

The last integral is equal to $-d\mathcal{E}_p/dt$, so that,

$$\frac{d\mathcal{E}_p}{dt} = - \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} \, dV. \quad (3.3.13)$$

The rate of change of the potential energy is positive when a point particle moves against the acceleration of gravity.

Next, we use the divergence theorem to convert the first integral on the right-hand side of (3.3.12) into a surface integral. Rearranging the resulting expression, we derive a mechanical energy balance expressed by the equation

$$\frac{d(\mathcal{E}_k + \mathcal{E}_p)}{dt} = \iint_{\text{parcel}} \mathbf{f} \cdot \mathbf{u} \, dS - \iiint_{\text{parcel}} \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV, \quad (3.3.14)$$

where $\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\sigma}$ is the traction and \mathbf{n} is the normal unit vector pointing *outward* from the parcel.

The first integral on the right-hand side of (3.3.14) is the rate of working of the traction on the parcel surface. It then follows from the first law of thermodynamic that the second integral on the right-hand side expresses the rate of energy production, denoted by $\dot{\mathcal{E}}$, causing a increase in the temperature of the fluid,

$$\dot{\mathcal{E}} = \iiint_{\text{parcel}} \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV, \quad (3.3.15)$$

consistent with the third term on the right-hand side of (3.3.6) pertaining to a control volume.

Parcel total energy balance

Part of the energy generated inside a parcel due to the fluid motion escapes by conduction across the parcel surface, and the remainder increases the internal energy of the fluid residing inside the parcel, \mathcal{E}_i , consistent with the second law of thermodynamics. We write

$$\frac{d\mathcal{E}_i}{dt} = Q_{\text{conduction}} + \dot{\mathcal{E}} + \dot{\mathcal{Q}}, \quad (3.3.16)$$

where $Q_{\text{conduction}}$ is the rate of heat conduction across the parcel surface and a $\dot{\mathcal{Q}}$ is the rate of interior heat production due, for example, a chemical reaction. Introducing the conductive flux, \mathbf{q} , we write

$$Q_{\text{conduction}} = - \iint_{\text{parcel}} \mathbf{q} \cdot \mathbf{n} \, dS \quad (3.3.17)$$

and rearrange (3.3.16) to obtain

$$\dot{\mathcal{E}} = \iint_{\text{parcel}} \mathbf{q} \cdot \mathbf{n} \, dS + \frac{d\mathcal{E}_i}{dt} - \dot{\mathcal{Q}}, \quad (3.3.18)$$

where \mathbf{n} is the normal unit vector pointing outward from the parcel. Combining this expression with (3.3.14), we obtain

$$\frac{d(\mathcal{E}_k + \mathcal{E}_p + \mathcal{E}_i)}{dt} = - \iint_{\text{parcel}} \mathbf{q} \cdot \mathbf{n} \, dS + \iint_{\text{parcel}} \mathbf{f} \cdot \mathbf{u} \, dS + \dot{\mathcal{Q}}, \quad (3.3.19)$$

which should be regarded as an evolution equation for the total energy of the parcel. *Note that, in the absence of interior heat production, surface integrals only appear on the right-hand side of this equation, that is, the total energy of a fluid parcel can change only by way of transport across the parcel surface.*

3.3.2 Internal energy transport

The internal energy of a fluid parcel can be expressed in terms of the specific internal energy, v , defined with respect to the density, ρ , as

$$\mathcal{E}_i = \iiint_{\text{parcel}} \rho v \, dV. \quad (3.3.20)$$

The rate of interior heat production can be expressed as a volume integral over a parcel volume,

$$\dot{Q} = \iiint_{\text{parcel}} \dot{q} \, dV. \quad (3.3.21)$$

where \dot{q} is the local rate of interior heat production with units of energy over volume divided by time elapsed.

To derive a differential total energy balance, we apply the divergence theorem to convert the surface integrals into volume integrals over the parcel on the right-hand side of (3.3.19). Noting that the volume of the parcel is arbitrary and using the continuity equation, we obtain

$$\rho \frac{D}{Dt} \left(\frac{1}{2} |\mathbf{u}|^2 + \mathbf{g} \cdot \mathbf{X} + v \right) = -\nabla \cdot \mathbf{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) + \dot{q}. \quad (3.3.22)$$

The three terms on the right-hand side express the effect of conduction, the rate of working of the traction, and the rate of internal energy production.

Evolution of kinetic and internal energy combined

Next, we apply the Reynolds transport theorem stated in (1.4.12) for the property field $\mathcal{P} = \rho \varepsilon$, where

$$\varepsilon \equiv \frac{1}{2} |\mathbf{u}|^2 + v, \quad (3.3.23)$$

and obtain

$$\frac{d(\mathcal{E}_k + \mathcal{E}_i)}{dt} = \iiint_{\text{parcel}} \frac{\partial(\rho \varepsilon)}{\partial t} \, dV + \iint_{\text{parcel}} \rho \varepsilon \mathbf{u} \cdot \mathbf{n} \, dS, \quad (3.3.24)$$

where \mathbf{n} is the outward unit normal vector. Substituting (3.3.24) and (3.3.13) into (3.3.19), and rearranging, we obtain a macroscopic energy balance,

$$\begin{aligned} & \iiint_{\text{parcel}} \frac{\partial(\rho \varepsilon)}{\partial t} \, dV + \iint_{\text{parcel}} \rho \varepsilon \mathbf{u} \cdot \mathbf{n} \, dS \\ &= - \iint_{\text{parcel}} \mathbf{q} \cdot \mathbf{n} \, dS + \iint_{\text{parcel}} \mathbf{f} \cdot \mathbf{u} \, dS + \iint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} \, dV + \dot{Q}. \end{aligned} \quad (3.3.25)$$

To derive the associated differential balance, we apply the divergence theorem to convert the surface integrals into volume integrals over the parcel. Noting that the volume of the parcel is arbitrary, we obtain

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \nabla \cdot (\rho\varepsilon \mathbf{u}) = -\nabla \cdot \mathbf{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) + \rho \mathbf{u} \cdot \mathbf{g} + \dot{q}. \quad (3.3.26)$$

This equation provides us with a vehicle for deriving an evolution equation for the specific internal energy, v .

Evolution of the specific internal energy

Now subtracting equation (3.3.26) from (3.3.4), we obtain the specific internal energy transport equation

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{u}) = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u} + \dot{q}. \quad (3.3.27)$$

Using the continuity equation, we find that, in terms of the material derivative,

$$\rho \frac{Dv}{Dt} = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u} + \dot{q}. \quad (3.3.28)$$

Integrating (3.3.27) over a parcel volume and applying the divergence theorem, we obtain

$$\frac{d}{dt} \iiint_{\text{parcel}} \rho v \, dV = - \iint_{\text{parcel}} \mathbf{q} \cdot \mathbf{n} \, dS + \iiint_{\text{parcel}} \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV + \dot{Q}, \quad (3.3.29)$$

where \mathbf{n} is the outward unit vector. *We have found that the internal energy of a parcel changes because of heat conduction across the parcel surface, heat production due to the fluid motion inside the particle volume, and interior heat production due, for example, to a chemical reaction.*

3.3.3 Enthalpy transport

It is useful to introduce the specific enthalpy, η , defined with respect to the fluid density, ρ , specific internal energy, v , and pressure, p , as

$$v = \eta - \frac{p}{\rho}. \quad (3.3.30)$$

Substituting this expression into (3.3.28), we obtain

$$\rho \frac{D\eta}{Dt} - \rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u} + \dot{q}, \quad (3.3.31)$$

where D/Dt is the material derivative.

Let $f(\rho)$ be an arbitrary function of the density, ρ , and ϕ be an arbitrary scalar field. Invoking the definition of the material derivative, we write

$$\frac{D(f\phi)}{Dt} \equiv \frac{\partial(f\phi)}{\partial t} + \mathbf{u} \cdot \nabla(f\phi) = f(\rho) \frac{D\phi}{Dt} + \phi f'(\rho) \frac{D\rho}{Dt}. \quad (3.3.32)$$

Using the continuity equation, we set $D\rho/Dt = -\rho \nabla \cdot \mathbf{u}$ and obtain

$$\frac{D(f\phi)}{Dt} = f(\rho) \frac{D\phi}{Dt} - \phi \rho f'(\rho) \nabla \cdot \mathbf{u}. \quad (3.3.33)$$

Now choosing $f(\rho) = 1/\rho$, we obtain

$$\rho \frac{D}{Dt} \left(\frac{\phi}{\rho} \right) = \frac{D\phi}{Dt} + \phi \nabla \cdot \mathbf{u}. \quad (3.3.34)$$

For $\phi = p$, we obtain

$$\rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) = \frac{Dp}{Dt} + p \nabla \cdot \mathbf{u}. \quad (3.3.35)$$

Substituting the right-hand side into the second term on the left-hand side of (3.3.31) and rearranging, we obtain

$$\rho \frac{D\eta}{Dt} = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u} + p \nabla \cdot \mathbf{u} + \frac{Dp}{Dt} + \dot{q}. \quad (3.3.36)$$

The second and third terms on the right-hand side combine to yield

$$\rho \frac{D\eta}{Dt} = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \frac{Dp}{Dt} + \dot{q}, \quad (3.3.37)$$

where

$$\boldsymbol{\sigma}^{\text{dev}} \equiv p \mathbf{I} + \boldsymbol{\sigma} \quad (3.3.38)$$

is the deviatoric stress tensor, sometimes also called the viscous stress tensor, and \mathbf{I} is the identity matrix. Relations from thermodynamics must be introduced at this point to relate the specific enthalpy to the fluid temperature and pressure.

3.3.4 Evolution of the temperature

The rate of change of the specific enthalpy is related to that of the absolute temperature, T , and pressure, p , by the equation

$$\frac{D\eta}{Dt} = c_p \frac{DT}{Dt} + \frac{1}{\rho} (1 - \alpha T) \frac{Dp}{Dt}, \quad (3.3.39)$$

where c_p is the heat capacity under constant pressure and

$$\alpha \equiv -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \quad (3.3.40)$$

is the coefficient of thermal expansion with units of inverse temperature. In the case of an ideal gas, $\alpha T = 1$.

Substituting (3.3.39) into (3.3.37) and rearranging, we obtain an evolution equation for the temperature,

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + p \nabla \cdot \mathbf{u} + \alpha T \frac{Dp}{Dt} + \dot{q}. \quad (3.3.41)$$

The rate of change of the pressure following a point particle must be simultaneously computed or otherwise specified.

3.3.5 Entropy transport

Gibbs' equation of thermodynamics states that

$$T \frac{Ds}{Dt} = \frac{Dv}{Dt} + p \frac{D}{Dt} \left(\frac{1}{\rho} \right), \quad (3.3.42)$$

where T is the absolute temperature and s is the specific entropy. Carrying out the differentiation and rearranging, we obtain

$$\frac{Dv}{Dt} = T \frac{Ds}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt}. \quad (3.3.43)$$

Substituting the right-hand side into the left-hand side of (3.3.28), invoking the continuity equation to set $D\rho/Dt = -\rho \nabla \cdot \mathbf{u}$, and rearranging, we obtain

$$\rho \frac{Ds}{Dt} = \frac{1}{T} (-\nabla \cdot \mathbf{q} + \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \dot{q}). \quad (3.3.44)$$

It is important to note that this equation is merely a restatement of energy conservation for a compressible or incompressible fluid.

Rearranging the first term on the right-hand side of (3.3.44), we obtain

$$\rho \frac{Ds}{Dt} = -\nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{1}{T^2} \mathbf{q} \cdot \nabla T + \frac{1}{T} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \frac{1}{T} \dot{q}. \quad (3.3.45)$$

Now integrating this equation over a parcel volume and applying the divergence theorem, we obtain the entropy transport equation

$$\begin{aligned} \frac{d}{dt} \iiint_{\text{parcel}} \rho s \, dV &= - \iint_{\text{parcel}} \frac{1}{T} \mathbf{q} \cdot \mathbf{n} \, dS \\ &\quad - \iiint_{\text{parcel}} \frac{1}{T^2} \mathbf{q} \cdot \nabla T \, dV + \iiint_{\text{parcel}} \frac{1}{T} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \, dV + \iiint_{\text{parcel}} \frac{1}{T} \dot{q} \, dV, \end{aligned} \quad (3.3.46)$$

where \mathbf{n} is the outward unit vector. The last integral on the right-hand side expresses the effect of fluid motion.

Clausius–Duhem inequality

The second law of thermodynamics specifies that

$$\frac{d}{dt} \iiint_{\text{parcel}} \rho s \, dV \geq - \iint_{\text{parcel}} \frac{1}{T} \mathbf{q} \cdot \mathbf{n} \, dS. \quad (3.3.47)$$

Combining this inequality with equation (3.3.46), we obtain the *Clausius–Duhem inequality*

$$-\frac{1}{T} \mathbf{q} \cdot \nabla T + \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \dot{q} \geq 0 \quad (3.3.48)$$

at every point in the fluid.

In the event that the stress tensor is symmetric, we obtain

$$-\frac{1}{T} \mathbf{q} \cdot \nabla T + \boldsymbol{\sigma}^{\text{dev}} : \widehat{\nabla \mathbf{u}} + \dot{q} \geq 0, \quad (3.3.49)$$

where $\widehat{\nabla \mathbf{u}}$ is the symmetric part of the velocity gradient tensor.

If the motion of the fluid does not affect the conductive transport, we obtain a pair of inequalities pertinent to heat transport and hydrodynamic energy production, respectively,

$$-\mathbf{q} \cdot \nabla T \geq 0, \quad \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \geq 0. \quad (3.3.50)$$

The first inequality requires that the flux due to a temperature gradient may not have a component against the temperature gradient.

3.3.6 Heat conduction

The generalized Fourier law is a *phenomenological* equation stipulating that the thermal flux at a point is related to the local temperature gradient, ∇T , by the equation

$$\mathbf{q} = -\mathbf{K} \cdot \nabla T, \quad (3.3.51)$$

where \mathbf{K} is the thermal conductivity tensor.

Equation (3.3.51) allows for the possibility that a temperature gradient in a certain direction may induce a flux in some other direction. In the case of an isotropic medium, $\mathbf{K} = k \mathbf{I}$, where k is the scalar thermal conductivity and \mathbf{I} is the identity matrix. In medical imaging technology (MRI), \mathbf{K} is known as the diffusion tensor.

The first inequality in (3.3.50) becomes

$$\mathbf{K} : \mathcal{T} \geq 0 \quad (3.3.52)$$

or

$$\hat{\mathbf{K}} : \mathcal{T} \geq 0, \quad (3.3.53)$$

where $\mathcal{T} \equiv \nabla T \otimes \nabla T$ is a symmetric matrix and a caret denotes the symmetric part. The last inequality requires that $\hat{\mathbf{K}}$ should be positive semi-definite.

In fact, it is often alleged in the literature that Onsager's principle implies that \mathbf{K} is symmetric and positive-definite. This means that three mutually orthogonal directions can always be found where the local flux is parallel to the local temperature gradient.

3.3.7 Transport processes

In Section 1.4, we discussed properties of fluid parcels in terms of the labeling Lagrangian vector field, $\boldsymbol{\alpha}$, including the parcel volume, mass, momentum, and angular momentum. Earlier in this section, we discussed further parcel properties including the kinetic, potential, and internal energies, the enthalpy, and the entropy.

A summary of parcel property evolution laws along with their physical origin and associated evolution differential equations, is given in Table 3.3.1, where \mathbf{n} is the *outward* unit vector normal to the parcel surface. Integral balances over a control volume, \mathcal{V}_c , bounded by a surface, or a collection of surfaces, \mathcal{D} are also given in the table, where \mathbf{n}^{in} is the inward normal unit vector, as shown in Figure 3.2.1. The integral balances arise by integrating the evolution equations over the control volume and then applying the Gauss divergence theorem.

A typical transport equation or integral balance has the typical form:

$$\begin{aligned} (\text{rate of accumulation}) &= (\text{rate of inward convection}) \\ &= \begin{pmatrix} \text{surface force} \\ \text{inward diffusion} \\ \vdots \end{pmatrix} + \begin{pmatrix} \text{volumetric production} \\ \text{body force} \\ \vdots \end{pmatrix}, \end{aligned}$$

where a choice is made on the appropriate term inside each set of parentheses on the right-hand side.

Problems

3.3.1 Eigenvalues and eigenvectors of the conductivity tensor

Consider a two-dimensional material with conductivity matrix

$$\mathbf{K} = c \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}, \quad (3.3.54)$$

in the square $-b < x < b$ and $-b < y < b$, where b is a given length and c is a constant. Prepare a three-dimensional plot of the first eigenvalue, a three-dimensional plot of the

Volume:	Kinematics:
$V_p \equiv \iiint dV$	$\frac{dV_p}{dt} = \iint \mathbf{u} \cdot \mathbf{n} dS$
	Jacobian-metric of Lagrangian mapping (\mathcal{J}) evolution equation:
	$\frac{D\mathcal{J}}{Dt} = \mathcal{J} \nabla \cdot \mathbf{u}, \quad \frac{\partial \mathcal{J}}{\partial t} + \nabla \cdot (\mathcal{J} \mathbf{u}) = 2 \mathcal{J} \nabla \cdot \mathbf{u}$
	Integral Jacobian-metric balance:
	$\iiint_{V_c} \frac{\partial \mathcal{J}}{\partial t} dV = \iint_{\mathcal{D}} \mathcal{J} \mathbf{u} \cdot \mathbf{n}^{\text{in}} dS + 2 \iiint_{V_c} \mathcal{J} \nabla \cdot \mathbf{u} dV$
Mass:	Mass conservation:
$m_p \equiv \iiint \rho dV$	$\frac{dm_p}{dt} = 0$
	Continuity equation:
	$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$
	Integral mass balance:
	$\iiint_{V_c} \frac{\partial \rho}{\partial t} dV = \iint_{\mathcal{D}} \rho \mathbf{u} \cdot \mathbf{n}^{\text{in}} dS$

TABLE 3.3.1 Properties of fluid parcels and associated rates of change dictated by physical laws or principles of thermodynamics. where \mathbf{q} is the conductive thermal flux; \dot{Q} is the rate of a parcel's interior heat production due for example, to chemical reaction; \dot{q} is the local rate of a parcel's interior heat production due for example, to chemical reaction. Applying the Reynolds transport theorem to the rate-of-change integrals provides us with balance equations. *Continuing* \rightarrow

Momentum:	Newton's law of motion:
$\mathbf{M}_p \equiv \iiint \rho \mathbf{u} \, dV$	$\frac{d\mathbf{M}_p}{dt} = \iint \mathbf{n} \cdot \boldsymbol{\sigma} \, dS + \iiint \rho \mathbf{g} \, dV$
	Cauchy equation of motion:
	$\rho \frac{D\mathbf{u}}{Dt} = \frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}$
	Integral momentum balance:
	$\iiint_{\mathcal{V}_c} \frac{\partial(\rho\mathbf{u})}{\partial t} \, dV = \iint_{\mathcal{D}} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) \, dS$ $- \iint_{\mathcal{D}} \mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma} \, dS + \iiint_{\mathcal{V}_c} \rho \mathbf{g} \, dV$
Angular momentum:	Newton's law of motion:
$\mathbf{A}_p \equiv \iiint \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{u} \, dV$	$\frac{d\mathbf{A}_p}{dt} = \iint (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{n} \cdot \boldsymbol{\sigma}) \, dS$ $+ \iiint \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g} \, dV$
	Cauchy equation of motion:
	$\rho (\mathbf{x} - \mathbf{x}_0) \times \frac{D\mathbf{u}}{Dt}$ $= \frac{\partial(\rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{u})}{\partial t} + \nabla \cdot (\rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{u} \otimes \mathbf{u})$ $= (\mathbf{x} - \mathbf{x}_0) \times \nabla \cdot \boldsymbol{\sigma} + \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}$
	Integral angular momentum balance:
	$\iiint_{\mathcal{V}_c} \frac{\partial(\rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{u})}{\partial t} \, dV$ $= \iint_{\mathcal{D}} \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{u} (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) \, dS$ $- \iint_{\mathcal{D}} (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma}) \, dS + \iiint_{\mathcal{V}_c} \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g} \, dV$

TABLE 3.3.1 \rightarrow Continued \rightarrow

<p>Kinetic energy:</p> $\mathcal{E}_k = \frac{1}{2} \iiint \rho \mathbf{u} ^2 dV$	<p>Cauchy equation</p> $\frac{d\mathcal{E}_k}{dt} = \iiint_{\text{parcel}} \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) dV - \iiint_{\text{parcel}} \boldsymbol{\sigma} : \nabla \mathbf{u} dV + \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} dV$ $\rho \frac{D(\frac{1}{2} \mathbf{u} ^2)}{Dt} = \frac{\partial \rho (\frac{1}{2} \mathbf{u} ^2)}{\partial t} + \nabla \cdot (\rho \frac{1}{2} \mathbf{u} ^2 \mathbf{u}) = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) - \boldsymbol{\sigma} : \nabla \mathbf{u} + \rho \mathbf{u} \cdot \mathbf{g}$ <p>Integral kinetic energy balance:</p> $\iiint_{\mathcal{V}_c} \frac{\partial (\rho (\frac{1}{2} \mathbf{u} ^2))}{\partial t} dV = \iint_{\mathcal{D}} \rho (\frac{1}{2} \mathbf{u} ^2) (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) dS = - \iint_{\mathcal{D}} (\mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} dS - \iiint_{\mathcal{V}_c} \boldsymbol{\sigma} : \nabla \mathbf{u} dV + \iiint_{\mathcal{V}_c} \rho \mathbf{u} \cdot \mathbf{g} dV$
<p>Potential energy:</p> $\mathcal{E}_p \equiv - \iiint \rho \mathbf{x} \cdot \mathbf{g} dV$	$\frac{d\mathcal{E}_p}{dt} = - \iiint_{\text{parcel}} \rho \mathbf{u} \cdot \mathbf{g} dV$
<p>Mechanical energy</p> $\mathcal{E}_m = \mathcal{E}_k + \mathcal{E}_p$ $\mathcal{E}_m = \iiint \rho (\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x}) dV$	<p>Cauchy equation:</p> $\frac{d(\mathcal{E}_k + \mathcal{E}_p)}{dt} = \iint (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} dS - \iiint \boldsymbol{\sigma} : \nabla \mathbf{u} dV$ $\rho \frac{D(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{X})}{Dt} = \frac{\partial (\rho (\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{X}))}{\partial t} + \nabla \cdot (\rho (\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{X}) \mathbf{u}) = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) - \boldsymbol{\sigma} : \nabla \mathbf{u}$ <p>Integral mechanical energy balance:</p> $\iiint_{\mathcal{V}_c} \frac{\partial (\rho (\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x}))}{\partial t} dV = \iint_{\mathcal{D}} \rho (\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x}) (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) dS = - \iint_{\mathcal{D}} (\mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} dS - \iiint_{\mathcal{V}_c} \boldsymbol{\sigma} : \nabla \mathbf{u} dV$

TABLE 3.3.1 → Continued →

<p>Internal energy</p> $\mathcal{E}_i = \iiint \rho v \, dV$	<p>Cauchy equation/Second law of thermodynamics:</p> $\frac{d\mathcal{E}_i}{dt} = - \iint \mathbf{n} \cdot \mathbf{q} \, dS + \iiint \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV + \dot{Q}$ $\rho \frac{Dv}{Dt} = \frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{u}) = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u}$ <p>Integral internal energy balance:</p> $\iiint_{\mathcal{V}_c} \frac{\partial(\rho v)}{\partial t} \, dV = \iint_{\mathcal{D}} \rho v \mathbf{u} \cdot \mathbf{n}^{\text{in}} \, dS$ $+ \iint_{\mathcal{D}} \mathbf{n}^{\text{in}} \cdot \mathbf{q} \, dS + \iiint_{\mathcal{V}_c} \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV$
<p>Total energy</p> $\mathcal{E}_t \equiv \mathcal{E}_k + \mathcal{E}_p + \mathcal{E}_i$ $\mathcal{E}_t = \iiint \rho \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right) \, dV$	<p>Cauchy equation/ Second law of thermodynamics:</p> $\frac{d\mathcal{E}_t}{dt} = - \iint \mathbf{n} \cdot \mathbf{q} \, dS + \iint \mathbf{u} \cdot (\mathbf{n} \cdot \boldsymbol{\sigma}) \, dS + \dot{Q}$ $\rho \frac{D \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right)}{Dt}$ $= \frac{\partial(\rho \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right))}{\partial t}$ $+ \nabla \cdot (\rho \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right) \mathbf{u})$ $= -\nabla \cdot \mathbf{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) + \dot{q}$ <p>Integral total energy balance:</p> $\iiint_{\mathcal{V}_c} \frac{\partial(\rho \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right))}{\partial t} \, dV$ $= \iint_{\mathcal{D}} \rho \left(\frac{1}{2} \mathbf{u} ^2 - \mathbf{g} \cdot \mathbf{x} + v \right) (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) \, dS$ $+ \iint_{\mathcal{D}} \mathbf{n}^{\text{in}} \cdot \mathbf{q} \, dS - \iint_{\mathcal{D}} (\mathbf{n}^{\text{in}} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} \, dS - \dot{Q}$

TABLE 3.3.1 → Continued →

Enthalpy	Cauchy equation / Second law of thermodynamics:
$H_p \equiv \iiint \rho \eta \, dV$	$\frac{dH_p}{dt} = - \iint \mathbf{q} \cdot \mathbf{n} \, dS + \iiint \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \, dV + \iiint \frac{Dp}{Dt} \, dV + \dot{Q}$
	$\rho \frac{D\eta}{Dt} = \frac{\partial(\rho\eta)}{\partial t} + \nabla \cdot (\rho\eta \mathbf{u}) = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \frac{Dp}{Dt} + \dot{q}$
	Integral enthalpy balance:
	$\begin{aligned} \iiint_{\mathcal{V}_c} \frac{\partial(\rho\eta)}{\partial t} \, dV &= \iint_{\mathcal{D}} \rho \eta (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) \, dS \\ &+ \iint_{\mathcal{D}} \mathbf{n}^{\text{in}} \cdot \mathbf{q} \, dS + \iiint_{\mathcal{V}_c} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \, dV + \iiint_{\mathcal{V}_c} \frac{Dp}{Dt} \, dV - \dot{q} \end{aligned}$
Entropy:	Cauchy equation / Second law of thermodynamics:
$S_p \equiv \iiint \rho s \, dV$	$\begin{aligned} \frac{dS_p}{dt} &= - \iint \frac{1}{T} \mathbf{q} \cdot \mathbf{n} \, dS - \iiint \frac{1}{T^2} \mathbf{q} \cdot \nabla T \, dV \\ &+ \iiint \frac{1}{T} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \, dV + \iiint \frac{1}{T} \dot{q} \, dV \end{aligned}$
	$\begin{aligned} \rho \frac{Ds}{Dt} &= \frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{u}) \\ &= -\nabla \cdot \left(\frac{1}{T} \mathbf{q} \right) - \frac{1}{T^2} \mathbf{q} \cdot \nabla T + \frac{1}{T} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} + \frac{1}{T} \dot{q} \end{aligned}$
	Integral entropy balance:
	$\begin{aligned} \iiint_{\mathcal{V}_c} \frac{\partial(\rho s)}{\partial t} \, dV &= \iint_{\mathcal{D}} \rho s (\mathbf{u} \cdot \mathbf{n}^{\text{in}}) \, dS + \iint_{\mathcal{D}} \frac{1}{T} \mathbf{n}^{\text{in}} \cdot \mathbf{q} \, dS \\ &- \iint_{\mathcal{V}_c} \frac{1}{T^2} \mathbf{q} \cdot \nabla T \, dS + \iiint_{\mathcal{V}_c} \frac{1}{T} \boldsymbol{\sigma}^{\text{dev}} : \nabla \mathbf{u} \, dV + \iiint_{\mathcal{V}_c} \frac{1}{T} \dot{q} \, dV \end{aligned}$

TABLE 3.3.1 → Continued.

second eigenvalue, a vector plot of the first eigenvector, and a vector plot of the second eigenvector.

3.3.2 Transport equation for the total energy

Derive the transport equation for the total energy, \mathcal{E}_t , shown in Table 3.3.1.

3.4 Constitutive equations for the stress tensor

Molecular motions in a fluid that has been in a macroscopic state of rest for a sufficiently long period of time reach dynamical equilibrium whereupon the stress field assumes the isotropic form

$$\boldsymbol{\sigma} = -p_{\text{th}} \mathbf{I}, \quad (3.4.1)$$

where \mathbf{I} is the identity matrix. The thermodynamic pressure, p_{th} , is a function of the density and temperature, and depends on the chemical composition of the fluid in a manner that is determined by an appropriate equation of state.

Ideal-gas law

In the case of an ideal gas, the thermodynamic pressure, p_{th} , is related to the density, ρ , by Clapeyron's ideal gas law,

$$p_{\text{th}} = \frac{RT}{M} \rho, \quad (3.4.2)$$

where $R = 8.314 \times 10^3 \text{ kg m}^2/(\text{s}^2 \text{ kmole K})$ is the ideal gas constant, T is Kelvin's absolute temperature, which is equal to the Celsius centigrade temperature reduced by 273 units, M is the molar mass, defined as the mass of one mole comprised of a collection of N_A molecules, and $N_A = 6.022 \times 10^{26}$ is the Avogadro number. The molar mass of an element is equal to the atomic weight of the element listed in the periodic table expressed in grams.

Effect of fluid motion

Physical intuition suggests that the instantaneous structure of the stress field inside a fluid that has been in a state of motion for some time depends not only on the current thermodynamic conditions, but also on the history of the motion of all point particles comprising the fluid, from inception of the motion, up to the present time. Leaving aside physiochemical interactions that are independent of the fluid motion, we argue that the stress field depends on the structure of the velocity field at all prior times.

This line of reasoning leads us to introducing a constitutive equation for the stress tensor that relates the stress at a point at a particular instant $t = \tau$ to the structure of the velocity field at all prior times,

$$\boldsymbol{\sigma}(t = \tau) = \mathcal{G}[\mathbf{u}(t \leq \tau)]. \quad (3.4.3)$$