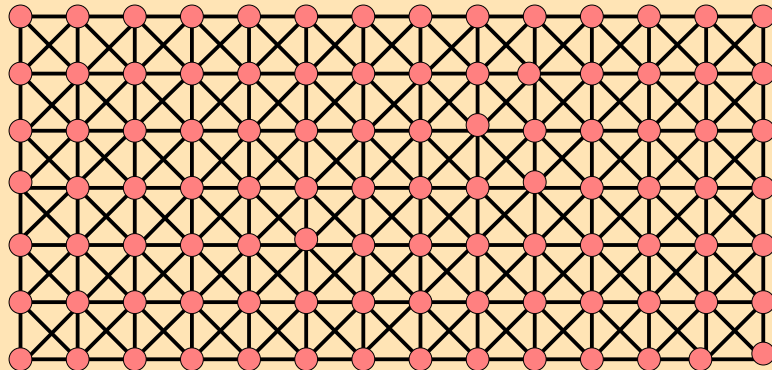


Introduction to Grids, Graphs, and Networks

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distributed sinks are available in the case of isolated networks. Discrete Green's functions can be used as building blocks for computing general solutions subject to given constraints.

This book is suitable for self-study and as a text in an upper-level undergraduate or entry-level graduate course in sciences, engineering, and applied mathematics. The material serves as a reference of terms, concepts, and as a resource of topics for further study.

C. Pozrikidis

Summer, 2013

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