

Contents

Preface	xi
Notation	xv
1 The fractional Laplacian in one dimension	1
1.1 Random walkers with constant steps	1
1.1.1 Particle number density distribution	2
1.1.2 Numerical simulation	4
1.2 Ordinary diffusion	9
1.2.1 Evolution of the variance	10
1.2.2 Population balance	11
1.3 Random jumpers	12
1.4 Central limit theorem and stable distributions	14
1.4.1 Ordinary diffusion	15
1.4.2 Anomalous diffusion	15
1.4.3 Continuous-time random walks (CTRW)	16
1.5 Power-law probability jump lengths	16
1.6 A principal-value integral	18
1.6.1 Regularization by singularity subtraction	19
1.6.2 Regularization by second-order differences	21
1.6.3 Evolution equations	21
1.7 Wires and springs	22
1.8 The fractional Laplacian	23
1.8.1 Definition in terms of a principal-value integral	23
1.8.2 Definition in terms of a regularized integral	24
1.8.3 Evaluation of the coefficient $c_{1,\alpha}$	25
1.8.4 Unsteady fractional diffusion equation	26
1.9 Fourier transform	27
1.9.1 The fractional Laplacian as a Fourier integral	28
1.9.2 Gaussian distribution	29
1.9.3 Computation of a degenerate hypergeometric function	30
1.10 Effect of fractional order	30
1.10.1 Gaussian distribution	31

1.10.2	Brinkman's approximation	32
1.11	Numerical computation of the fractional Laplacian	34
1.12	Green's function of the fractional Laplace equation	36
1.12.1	Solution by the Fourier transform	36
1.12.2	The Riesz potential	38
1.12.3	Fractional Laplacian as a self-induced Riesz potential	39
1.12.4	Inverse of the fractional Laplacian	40
1.13	Fractional Poisson equation in a restricted domain	40
1.13.1	Homogeneous extended Dirichlet boundary condition	41
1.13.2	Arbitrary extended Dirichlet boundary condition	46
1.13.3	Periodicity condition	47
1.14	Green's function of unsteady fractional diffusion	49
1.14.1	Solution by the Fourier transform	49
1.14.2	Evolution of the variance	51
1.14.3	Solution of the initial-value problem	51
1.14.4	Brinkman's approximation	52
2	Numerical discretization in one dimension	53
2.1	Computation of a principal-value integral	53
2.1.1	Mid-point integration rule	54
2.1.2	Influence coefficients	56
2.1.3	Infinite discretization	59
2.2	Fractional Laplacian differentiation matrix	60
2.2.1	Infinite discretization	63
2.2.2	Brinkman's approximation	63
2.3	Fractional Poisson equation	65
2.3.1	Homogeneous extended Dirichlet boundary condition	65
2.3.2	Arbitrary extended Dirichlet boundary conditions	66
2.4	Evolution under fractional diffusion	68
2.4.1	Implicit time integration	69
2.4.2	Evolution of the Gaussian distribution	70
2.4.3	Arbitrary extended Dirichlet boundary condition	73
2.4.4	Jumping creatures	73
2.4.5	Denoising	75
2.4.6	Fisher's equation	75
2.4.7	Numerical stability	76
2.4.8	Advanced time-integration schemes	78
2.5	Differentiation by spectral expansion	79
2.5.1	Infinite discretization	81
2.5.2	Boundary conditions	84
3	Further concepts in one dimension	85
3.1	Fractional first derivative	85
3.1.1	Definite integral of the fractional Laplacian	87

3.1.2	Effect of fractional order	88
3.1.3	Gaussian distribution	88
3.1.4	Numerical evaluation	90
3.2	Properties of the fractional first derivative	90
3.2.1	Even functions	92
3.2.2	Odd functions	93
3.2.3	Arbitrary functions	93
3.2.4	Fractional diffusive flux	94
3.3	The Laplacian potential	96
3.3.1	Numerical evaluation	98
3.3.2	The Laplacian potential in terms of the Riesz potential	99
3.4	Fractional derivatives from finite-difference stencils	101
3.4.1	Second-order finite-difference stencils	101
3.4.2	Fourth-order finite-difference stencils	103
3.5	Fractional third derivative	104
3.5.1	Fourier transform	105
3.5.2	Numerical evaluation	106
3.6	Fractional fourth derivative	108
3.6.1	Fourier transform	109
3.6.2	Numerical evaluation	110
4	Periodic functions	115
4.1	Sine, cosines, and the complete Fourier series	115
4.1.1	Complete Fourier series	116
4.1.2	Square wave	117
4.2	Cosine Fourier series	119
4.3	Sine Fourier series	122
4.3.1	Numerical evaluation of the Fourier coefficients	124
4.3.2	Fractional Laplacian	125
4.3.3	Poisson equation	126
4.4	Green's functions	128
4.4.1	Solution of the Poisson equation	129
4.4.2	Periodic Moore–Penrose Green's function	130
4.5	Integral representation of the periodic Laplacian	131
4.5.1	Representation as a regularized integral	133
4.6	Numerical discretization	134
4.7	Periodic differentiation matrix	137
4.7.1	Fractional Poisson equation	139
4.7.2	Evolution under fractional diffusion	142
4.8	Differentiation by spectral expansion	143
4.8.1	Fractional differentiation matrix	144
4.8.2	Fractional graph Laplacian	145
4.9	Embedding of the fractional Poisson equation	148

5	The fractional Laplacian in three dimensions	153
5.1	Stipulation on the Fourier transform	153
5.2	Integral representation	155
5.2.1	Notion of the principal-value integral	155
5.2.2	Local contribution	156
5.2.3	Regularized integral representations	157
5.2.4	Determination of the coefficient $c_{3,\alpha}$	158
5.3	Fractional gradient	160
5.3.1	Integral representation	161
5.3.2	Reduction to the ordinary gradient	164
5.3.3	Numerical evaluation	165
5.3.4	Fractional diffusive flux	166
5.4	Laplacian potential	168
5.5	Green's function of the fractional Laplace equation	169
5.6	The Riesz potential	173
5.6.1	Inverse of the fractional Laplacian	174
5.6.2	Laplacian potential	174
5.7	Triply periodic Green's function	175
5.7.1	Green's function as a Fourier series	177
5.7.2	Ewald summation	177
5.8	Fractional Poisson equation	183
5.8.1	Exact solution in a sphere	183
5.8.2	Numerical methods	184
5.9	Evolution under fractional diffusion	187
5.10	Periodic functions and arbitrary domains	189
5.11	Fractional Stokes flow	190
5.11.1	Equations of fractional Stokes flow	191
5.11.2	Flow due to a point force	192
5.11.3	Fractional Stokeslet	193
6	The fractional Laplacian in two dimensions	199
6.1	Stipulation on the Fourier transform	199
6.2	Integral representation	200
6.2.1	Notion of the principal-value integral	200
6.2.2	Local contribution	200
6.2.3	Regularized integral representations	202
6.2.4	Determination of the coefficient $c_{2,\alpha}$	202
6.3	Fractional gradient	205
6.3.1	Integral representation	205
6.3.2	Reduction to the ordinary gradient	206
6.3.3	Numerical evaluation	207
6.4	Laplacian potential	208

6.5	Green's function of the fractional Laplace equation	209
6.6	The Riesz potential	212
6.6.1	Inverse of the fractional Laplacian	213
6.6.2	Laplacian potential	213
6.7	Doubly periodic Green's function	214
6.7.1	Green's function as a Fourier series	215
6.7.2	Ewald summation	216
6.8	Fractional Poisson equation	219
6.8.1	Exact solution inside a circular disk	219
6.8.2	Numerical methods	221
6.9	Evolution due to fractional diffusion	227
6.10	Periodic functions and arbitrary domains	227
A	Selected definite integrals	231
B	The Gamma function	235
C	The Gaussian distribution	239
D	The fractional Laplacian in arbitrary dimensions	241
D.1	Fourier transform	241
D.2	Integral representation of the fractional Laplacian	242
D.3	Fractional gradient	244
D.4	Green's function of the fractional Laplace equation	245
D.5	The Riesz potential	246
D.5.1	Inverse of the fractional Laplacian	246
D.5.2	Laplacian potential	247
D.6	Fractional Poisson equation in a finite solution domain	247
D.6.1	Homogeneous extended Dirichlet boundary condition	248
D.7	Green's function of unsteady fractional diffusion	250
D.7.1	One dimension	253
D.7.2	Two dimensions	254
D.7.3	Three dimensions	254
D.7.4	Arbitrary dimensions	255
D.8	Embedding of the fractional Poisson equation	256
D.9	Fractional graph Laplacian	258
E	Fractional derivatives	259
E.1	The Riemann–Liouville fractional derivative	259
E.2	Grünwald–Letnikov derivative	260
E.3	Caputo fractional derivative	264
F	Aitken extrapolation of an infinite sum	267

F.1 Riemann's zeta function	269
F.2 Kernel of the periodic fractional Laplacian	271
References	273
Index	275