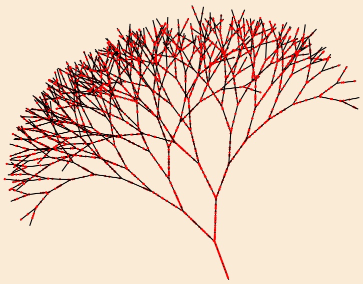
FDLIB

Fluid Dynamics Software Library

User Guide



Directory: `12_bem`

Boundary-element methods

Version 13.10

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http://dehesa.freeshell.org/FDLIB

Contents

Preface

FDLIB is a comprehensive software library of FORTRAN 77 (compatible with FORTRAN 90), Matlab, C++, and other codes, covering a broad spectrum of fundamental and applied topics in fluid dynamics. The codes are suitable for self-study, classroom instruction, and fundamental or applied research. The utility subroutines and simulation programs of FDLIB are pertinent to a variety of disciplines in science and engineering including applied mathematics and scientific computing, computational physics, biomechanics, aerospace engineering, mechanical engineering, chemical engineering, electrical engineering, and bioengineering.

This document contains the user guide of the twelfth directory of FDLIB concerning boundary-element methods.

Directory contents

12_bem

This directory contain codes that solve Laplace's equation subject to Dirichlet and Neumann boundary conditions, using boundary-element methods. The subdirectories are shown in the following table:

Subdirectory	Topic
ldr_3d	Solution of Laplace's equation in the interior or exterior of a three-dimensional surface, subject to the Dirichlet boundary condition. The solution is found using a boundary element method based on Green's third identity.
ldr_3d_2p	Solution of Laplace's equation in a semi-infinite domain subject to the Dirichlet boundary condition. The solution is found using a boundary-element method based on the double-layer representation.
ldr_3d_ext	Solution of Laplace's equation in the exterior of a three-dimensional surface, subject to the Dirichlet boundary condition. The solution is found using a boundary-element method based on the completed double-layer representation.
ldr_3d_int	Solution of Laplace's equation in the interior of a three-dimensional surface, subject to the Dirichlet boundary condition. The solution is found using a boundary-element method based on the double-layer representation.
lnm_3d	Solution of Laplace's equation in the interior or exterior of a three-dimensional surface, subject to the Neumann boundary condition. The solution is found using a boundary-element method based on Green's third identity.

Directory: ldr_3d

This directory contains a code that solves Laplace's equation in the interior or exterior of a closed three-dimensional surface, subject to the Dirichlet boundary condition that specifies the surface distribution of the unknown function. The problem is formulated as an integral equation of the first kind originating from Green's third identity. The mathematical formulation and numerical method are discussed in Reference [1].

References

[1] Pozrikidis, C. (2002) *A Practical Guide to Boundary Element Methods with the Software Library BEMLIB*, Chapman & Hall/CRC.

Directory: ldr_3d_2p

This directory contains a code that solves Laplace's equation above a semi-infinite periodic surface of a closed three-dimensional surface subject to the Dirichlet boundary condition that specifies the surface distribution of the unknown function, as illustrated in Figure ldr_3d_2p.1. The solution is found by solving an integral equation of the second kind originating from the double-layer representation, using a boundary-element method.

Mathematical formulation

We seek a scalar function, $f(\mathbf{x})$, defined above a doubly periodic surface D, satisfying Laplace's equation in three dimensions,

$$\nabla^2 f = 0, \tag{1}$$

subject to the Dirichlet boundary condition

$$f(\mathbf{x}) = \mathcal{F}(\mathbf{x}), \tag{2}$$

where the point $\mathbf{x}$ lies on D and $\mathcal{F}(\mathbf{x})$ is a specified surface function. The periodicity of the surface is defined by two base vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$.

In the mathematical formulation, the function f is represented by a double-layer harmonic potential defined over one period of the surface D,

$$f(\mathbf{x}_0) = \iint_D q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{3}$$

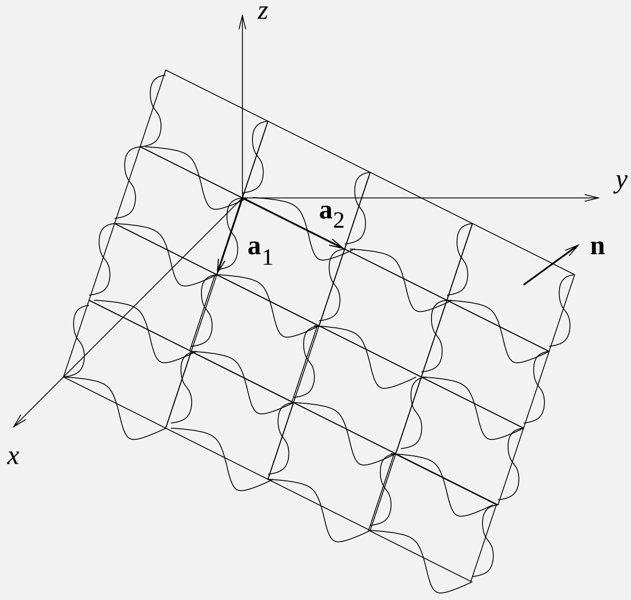

Figure ldr_3d_2p.1 Solution of Laplace's equation above a doubly periodic surface with arbitrary orientation subject to the Dirichlet boundary condition, computed by a boundary-element method based on the double-layer representation.

where q is the *a priori* unknown density of the double-layer potential, $\mathcal{G}(\mathbf{x}, \mathbf{x}_0)$ is doubly-periodic Green's function, and $\mathbf{n}$ is the unit vector normal to the surface pointing upward.

Taking the limit of the integral representation (3) as the point $\mathbf{x}_0$ approaches the surface D from above, and expressing the limit of the double-layer integral in terms of its principal value, we derive an integral equation of the second kind for the double-layer density, q,

$$\mathcal{F}(\mathbf{x}_0) = -\frac{1}{2}\, q(\mathbf{x}_0) + \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{4}$$

where PV denotes the principal-value integral. Rearranging (4), we obtain the standard integral equation form

$$q(\mathbf{x}_0) = 2 \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - 2\, \mathcal{F}(\mathbf{x}_0). \tag{5}$$

Representation in terms of a vector potential

Alternatively, the gradient of the double-layer potential can be expressed as the curl of the corresponding vector potential $\mathbf{A}$,

$$\nabla f(\mathbf{x}_0) = \nabla_0 \times \mathbf{A}(\mathbf{x}_0). \tag{6}$$

The vector potential is given by the integral representation

$$\mathbf{A}(\mathbf{x}_0) = \iint_D \boldsymbol{\zeta}(\mathbf{x})\, \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{7}$$

where $\boldsymbol{\zeta}$ is the strength of the vortex sheet associated with the double-layer potential, given by

$$\boldsymbol{\zeta} = \mathbf{n} \times \nabla q \tag{8}$$

(e.g., [1], p. 503). The right-hand side of (12) involves tangential derivatives of the density distribution q alone.

To compute the normal derivative $\mathbf{n} \cdot \nabla f$, we work in four stages:

- Solve the integral equation (8), and then recover the double-layer density q from (9).
- Compute the strength of the vortex sheet ζ from (12).
- Evaluate the distribution of the vector potential $\mathbf{A}$ over D from (11).
- Compute $\mathbf{n} \cdot \nabla f$ in terms of tangential derivatives of the vector potential.

Numerical method

One period of the surface, D, is discretized into an unstructured grid of six-node curved triangles. All geometrical variables and the unknown function are approximated with quadratic functions over each triangle in terms of local triangle (barycentric) coordinates. The integral equation (5) is solved by the method of successive substitutions.

The principal value of the double-layer integral on the right-hand side of (5) is computed using the identity

$$\iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) = \int_D [q(\mathbf{x}) - q(\mathbf{x}_0)\,]\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - \frac{1}{2}\, q(\mathbf{x}_0). \tag{9}$$

The non-singular integral on the right-hand side of (9) is computed by the Gauss triangle quadrature.

After the integral equation has been solved, the function f at a point in the solution domain (exterior of D) is evaluated from the integral representation (3).

Program files:

1. `cramer_33`
   Solution of a 3×3 linear system by Cramer's rule.
2. `gauss_leg`
   Base points and weights for the Gauss-Legendre quadrature.
3. `gauss_trgl`
   Base points and weights for integrating over a triangle.
4. `ldr_3d_2p`
   The main program solves the integral equation for the strength of the double-layer potential, computes the surface distribution of the normal derivative $\partial f/\partial n$, and evaluates the solution at a specified point using the integral representation.
5. `ldlp_3d_2p`
   Computation of the principal value of the double-layer potential at the grid nodes.
6. `ldlpp_3d_2p`
   Computation of the nonsingular double-layer potential at a point in the solution domain.
7. `ldr_3d_geo`
   Various auxiliary computations regarding geometry.
8. `lgf_3d_fs`
   Free-space Green's function of Laplace's equation in three dimensions.
9. `lslp3_3d_2p`
   Computation of the Laplace single-layer potential of a vector function at the nodes.
10. `sgrad_3d_2p`
    Computation of the surface gradient of a scalar function.
11. `srf_int_3d_2p`
    Computation of a surface integral of a scalar function.
12. `trgl_sqr`
    Discretization of a square patch a grid of six-node curved triangles.
13. `vs_3d_2p_circ_vort`
    Computation of the strength of the vortex sheet at the nodes in terms of tangential derivatives of the double-layer density (surface circulation).
14. `vs_3d_2p_curl`
    Computation of the component of the curl of a vector function normal to the vortex sheet.

Input file:

1. `ldr_3d_2p.dat`
   Specification of the problem parameters.

Output file:

1. `ldr_3d_2p.out`
   Computed values of the double-layer strength and normal derivative of the surface function f at the nodes.

References

[1] Pozrikidis, C. (2011) *Introduction to Theoretical and Computational Fluid Dynamics.* Second Edition, Oxford University Press.

Directory: ldr_3d_ext

This directory contains a code that solves Laplace's equation in the exterior of a closed three-dimensional surface, subject to the Dirichlet boundary condition that specifies the distribution of the unknown function on the surface, as illustrated in figure ldr_3d_ext.1. The problem is formulated as an integral equation of the second kind originating from the completed double-layer representation, using a boundary-element method.

Mathematical formulation

We want to compute a scalar function, $f(\mathbf{x})$, in the exterior of a closed surface D, satisfying Laplace's equation in three dimensions,

$$\nabla^2 f = 0, \tag{1}$$

subject to the Dirichlet boundary condition specifying that

$$f(\mathbf{x}) = \mathcal{F}(\mathbf{x}), \tag{2}$$

where the point $\mathbf{x}$ lies on D, and $\mathcal{F}(\mathbf{x})$ is a specified surface function.

In the mathematical formulation, the function f is represented by a combined double-layer harmonic potential defined over the surface D, and a point source situated at the surface centroid $\mathbf{x}_c$, as

$$f(\mathbf{x}_0) = \iint_D q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - s(q)\, \mathcal{G}(\mathbf{x}_0, \mathbf{x}_c), \tag{3}$$

where q is the *a priori* unknown density of the double-layer potential, $s(q)$ is the strength of the point source, $\mathcal{G}(\mathbf{x}, \mathbf{x}_0) = 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ is the free-space Green's function, $\mathbf{n}$ is the unit vector normal to the surface pointing into the *exterior*, S_D is the surface area of D (e.g., [1], p. 516).

To remove the ambiguity in the definition of the strength of the point source while preserving linearity, we require

$$s(q) \equiv -\beta \left(\frac{4\pi}{S_D}\right)^{1/2} \iint_D q\, \mathrm{d}S, \tag{4}$$

where β is an arbitrary dimensionless coefficient.

Taking the limit of the integral representation (2) as the point $\mathbf{x}_0$ approaches the surface D from the exterior, and expressing the limit of the double-layer integral in terms of its principal value, we obtain an integral equation of the second kind for the double-layer density, q,

$$\mathcal{F}(\mathbf{x}_0) = \frac{1}{2}\, q(\mathbf{x}_0) + \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - s(q)\, \mathcal{G}(\mathbf{x}_0, \mathbf{x}_c), \tag{5}$$

where PV denotes the principal value integral. Rearranging (5), we obtain a standard integral equation,

$$q(\mathbf{x}_0) = 2\, \mathcal{F}(\mathbf{x}_0) - 2 \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) + 2\, s(q)\, \mathcal{G}(\mathbf{x}_0, \mathbf{x}_c). \tag{6}$$

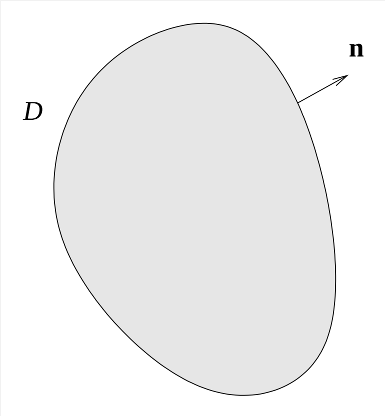

Figure ldr_3d_ext.1 Solution of Laplace's equation in the exterior of a three-dimensional surface, subject to the Dirichlet boundary condition, computed by a boundary element-method based on the completed double-layer representation.

Eigenvalue spectrum

The homogeneous equation corresponding to (6) is

$$\psi(\mathbf{x}_0) = -2 \iint_D^{PV} \psi(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) + 2\, s(\psi)\, \mathcal{G}(\mathbf{x}_0, \mathbf{x}_c), \tag{7}$$

where ψ is an eigenfunction. When the last term on the right-hand side is absent, any constant function is an eigensolution of (7), and the integral equation (5) does not have a unique solution. Consequently, the presence of the point source is imperative, in agreement with physical intuition.

The spectrum of eigenvalues of the compact integral operator expressed by the right-hand side of (7) is unknown. Insights can be gained by considering a spherical surface of radius a, placing the point source at the center, and setting the eigenfunction ψ equal to the constant value c. Using the definition of the point source strength s shown in (4), we find $s(c) = -\beta c 4\pi a$. Next, we substitute $G(\mathbf{x}_0, \mathbf{x}_c) = 1/(4\pi a)$ in (7) to obtain $c = (1 - 2\beta)c$, which shows that a constant eigenfunction exists only when $\beta = 0$. A necessary but not sufficient condition for the integral operator on the right-hand side of (7) to define a contraction mapping is $-1 < 1 - 2\beta < 1$ or $0 < \beta < 1$.

Representation in terms of a vector potential

Alternatively, the velocity field associated with the double-layer potential can be expressed as the curl of the corresponding vector potential $\mathbf{A}$, in the form

$$\mathbf{u}(\mathbf{x}_0) = \nabla_0 \times \mathbf{A}(\mathbf{x}_0) - s(q)\, \nabla G(\mathbf{x}_0, \mathbf{x}_c). \tag{8}$$

The vector potential is given by the integral representation

$$\mathbf{A}(\mathbf{x}_0) = \iint_D \boldsymbol{\zeta}(\mathbf{x})\, G(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{9}$$

where $\boldsymbol{\zeta}$ is the strength of the vortex sheet corresponding to the double-layer, given by

$$\boldsymbol{\zeta} = \mathbf{n} \times \nabla q \tag{10}$$

(e.g., [1], p. 503) The right-hand side of (10) involves tangential derivatives of the density distribution, $q(\mathbf{x})$.

To compute the normal derivative, $\mathbf{n} \cdot \nabla f$, we work in four stages:

- Solve the integral equation (6) for the double-layer density q,
- Compute the strength of the vortex sheet ζ from (10).
- Evaluate the distribution of the vector potential $\mathbf{A}$ over D from (9).
- Compute $\mathbf{n}\cdot\nabla f$ in terms of tangential derivatives of the vector potential and the normal component of the point source corresponding to the second term on the right-hand side of (3).

Numerical method

The surface D is discretized into an unstructured grid of six-node curved triangles. All geometrical variables and the unknown function are approximated with quadratic functions over each triangle in terms of local triangle (barycentric) coordinates. The integral equation (6) is solved by the method of successive substitutions.

The principal value of the double-layer integral on the right-hand side of (6) is computed using the identity

$$\iint_D^{PV} q(\mathbf{x})\,\mathbf{n}(\mathbf{x})\cdot\nabla\mathcal{G}(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}S(\mathbf{x}) = \iint_D \big[\,q(\mathbf{x})-q(\mathbf{x}_0)\,\big]\,\mathbf{n}(\mathbf{x})\cdot\nabla\mathcal{G}(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}S(\mathbf{x}) - \frac{1}{2}\,q(\mathbf{x}_0). \tag{11}$$

The non-singular integral on the right-hand side of (11) is computed by the Gauss triangle quadrature.

After the integral equation has been solved, the function f at a point in the solution domain (exterior of D) is computed from the integral representation (3).

Program files:

1. `cramer_33`
   Solution of a 3×3 linear system by Cramer's rule.
2. `gauss_leg`
   Base points and weights for the Gauss-Legendre quadrature.
3. `gauss_trgl`
   Base points and weights for integration over a triangle.
4. `ldlp_3d`
   Computation of the principal value of the double-layer potential at the grid nodes.
5. `ldlpp_3d`
   Computation of the nonsingular double-layer potential at a point in the solution domain.
6. texttldr_3d_ext
   The main program solves the integral equation (5) for the strength of the double-layer potential, computes the surface distribution of the normal derivative $\partial f/\partial n$, and evaluates the solution at a specified point using the integral representation.
7. `ldr_3d_geo`
   Various auxiliary computations regarding geometry.

8. `lgf_3d_fs`
   Free-space Green's function of Laplace's equation in three dimensions.

9. `lslp3_3d`
   Computation of the Laplace single-layer potential of a vector function at the nodes.

10. `sgrad_3d`
    Computation of the surface gradient of a scalar function.

11. `srf_int_3d`
    Computation of a surface integral of a scalar function.

12. `trgl_octa`
    Discretization of a closed surface into a grid of six-node curved triangles descending from the octahedron.

13. `vs_3d_circ_vort`
    Computation of the strength of the vortex sheet at the nodes in terms of tangential derivatives of the double-layer density (surface circulation).

14. `vs_3d_curl`
    Computation of the component of the curl of a vector function normal to the vortex sheet.

Input file:

1. `ldr_3d_ext.dat`
   Specification of the problem parameters.

Output file:

1. `ldr_3d_ext.out`
   Computed values of the double-layer strength and normal derivative of the surface function f at the nodes.

References

[1] Pozrikidis, C. (2011) *Introduction to Theoretical and Computational Fluid Dynamics.* Second Edition, Oxford University Press.

Directory: ldr_3d_int

This directory contains a code that solves Laplace's equation in the interior of a closed three-dimensional surface, subject to the Dirichlet boundary condition that specifies the distribution of the unknown function over the surface, as illustrated in figure ldr_3d_int.1. The solution is found by solving an integral equation of the second kind originating from the double-layer representation, using a boundary-element method.

Mathematical formulation

We are looking for a scalar function $f(\mathbf{x})$ in the interior of a closed surface D, satisfying Laplace's equation in three dimensions,

$$\nabla^2 f = 0, \tag{1}$$

subject to the Dirichlet boundary condition

$$f(\mathbf{x}) = \mathcal{F}(\mathbf{x}), \tag{2}$$

where the point $\mathbf{x}$ lies on D, and $\mathcal{F}(\mathbf{x})$ is a specified surface function.

In the mathematical formulation, the function f is represented by a double-layer harmonic potential defined over D,

$$f(\mathbf{x}_0) = \iint_D q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{3}$$

where q is the *a priori* unknown density of the double-layer potential, $\mathcal{G}(\mathbf{x}, \mathbf{x}_0) = 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ is the free-space Green's function, and $\mathbf{n}$ is the unit vector normal to the surface pointing into the *exterior* of D (e.g., [1], p. 516).

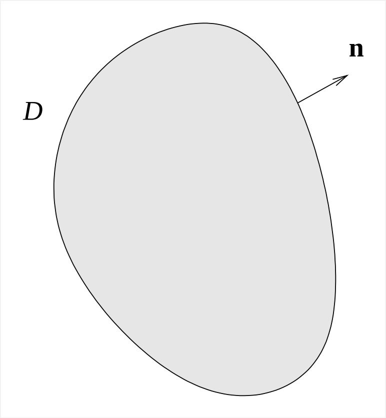

Figure ldr_3d_int.1 Solution of Laplace's equation in the interior of a three-dimensional surface subject to the Dirichlet boundary condition, computed by a boundary-element method based on the double-layer representation.

Taking the limit of the integral representation (3) as the point $\mathbf{x}_0$ approaches the surface D from the interior, and expressing the limit of the double-layer integral in terms of its principal value, we derive an integral equation of the second kind for the double-layer density, q,

$$\mathcal{F}(\mathbf{x}_0) = -\frac{1}{2}\, q(\mathbf{x}_0) + \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{4}$$

where PV denotes the principal-value integral. Rearranging (4), we obtain the standard integral equation form

$$q(\mathbf{x}_0) = 2 \iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - 2\, \mathcal{F}(\mathbf{x}_0). \tag{5}$$

The homogeneous equation corresponding to (5) arises by discarding the forcing function,

$$\psi(\mathbf{x}_0) = 2 \iint_D^{PV} \psi(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{6}$$

where ψ is an eigenfunction. It can be shown that the homogeneous equation does not admit a solution, and therefore equation (5) has a unique solution. The generalized homogeneous equation is

$$\psi(\mathbf{x}_0) = 2\lambda \iint_D^{PV} \psi(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}). \tag{7}$$

where λ is an eigenvalue. The set of eigenvalues comprises the spectrum of the double-layer potential. The spectrum is known to contain the eigenvalue, $\lambda = -1$, corresponding to a constant eigenfunction, and no other eigenvalue in the closed interval $[-1, 1]$. The presence of this marginal eigenvalue prevents us from solving the integral equation (5) by the method of successive substitutions.

To circumvent this difficulty, we replace equation (5) with the deflated integral equation

$$\hat{q}(\mathbf{x}_0) = 2 \iint_D^{PV} \hat{q}(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - 2\, \mathcal{F}(\mathbf{x}_0) + \frac{1}{S_D} \iint_D \hat{q}(\mathbf{x})\, \mathrm{d}S(\mathbf{x}), \tag{8}$$

where S_D is the surface area of D, and $\hat{q}$ is an auxiliary distribution. The double-layer density is given by

$$q = \hat{q} - \frac{1}{2\, S_D} \iint_D \hat{q}(\mathbf{x})\, \mathrm{d}S(\mathbf{x}). \tag{9}$$

It can be shown that the integral equation (8) can be solved by the method of successive substitutions. Once the solution has been found, the dipole strength, q, arises from (9).

Representation in terms of a vector potential

Alternatively, the gradient of the double-layer potential can be expressed as the curl of the corresponding vector potential $\mathbf{A}$, in the form

$$\nabla f(\mathbf{x}_0) = \nabla_0 \times \mathbf{A}(\mathbf{x}_0). \tag{10}$$

The vector potential is given by the integral representation

$$\mathbf{A}(\mathbf{x}_0) = \iint_D \boldsymbol{\zeta}(\mathbf{x})\, \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}), \tag{11}$$

where $\boldsymbol{\zeta}$ is the strength of the vortex sheet associated with the double-layer potential, given by

$$\boldsymbol{\zeta} = \mathbf{n} \times \nabla q \tag{12}$$

(e.g., [1], p. 503). The right-hand side of (12) involves tangential derivatives of the density distribution q alone.

To compute the normal derivative $\mathbf{n} \cdot \nabla f$, we work in four stages:

- Solve the integral equation (8), and then recover the double-layer density q from (9).
- Compute the strength of the vortex sheet ζ from (12).
- Evaluate the distribution of the vector potential $\mathbf{A}$ over D from (11).
- Compute $\mathbf{n} \cdot \nabla f$ in terms of tangential derivatives of the vector potential.

Numerical method

The surface, D, is discretized into an unstructured grid of six-node curved triangles. All geometrical variables and the unknown function are approximated with quadratic functions over each triangle in terms of local triangle (barycentric) coordinates. The integral equation (6) is solved by the method of successive substitutions.

The principal value of the double-layer integral on the right-hand side of (6) is computed using the identity

$$\iint_D^{PV} q(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) = \int_D [q(\mathbf{x}) - q(\mathbf{x}_0)\,]\, \mathbf{n}(\mathbf{x}) \cdot \nabla \mathcal{G}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x}) - \frac{1}{2}\, q(\mathbf{x}_0). \tag{13}$$

The non-singular integral on the right-hand side of (11) is computed by the Gauss triangle quadrature.

After the integral equation has been solved, the function f at a point in the solution domain (exterior of D) is evaluated from the integral representation (3).

Files to be linked:

1. `cramer_33`
   Solution of a 3×3 linear system by Cramer's rule.
2. `gauss_leg`
   Base points and weights for the Gauss-Legendre quadrature.
3. `gauss_trgl`
   Base points and weights for integrating over a triangle.
4. `ldlp_3d`
   Computation of the principal value of the double-layer potential at the grid nodes.
5. `ldlpp_3d`
   Computation of the nonsingular double-layer potential at a point in the solution domain.
6. `ldr_3d_int`
   The main program solves the integral equation for the strength of the double-layer potential, computes the surface distribution of the normal derivative $\partial f/\partial n$, and evaluates the solution at a specified point using the integral representation.
7. `ldr_3d_geo`
   Various auxiliary computations regarding geometry.
8. `lgf_3d_fs`
   Free-space Green's function of Laplace's equation in three dimensions.

9. `lslp3_3d`
   Computation of the Laplace single-layer potential of a vector function at the nodes.

10. `sgrad_3d`
    Computation of the surface gradient of a scalar function.

11. `srf_int_3d`
    Computation of a surface integral of a scalar function.

12. `trgl_octa`
    Discretization of a closed surface into a grid of six-node curved triangles descending from the octahedron.

13. `vs_3d_circ_vort`
    Computation of the strength of the vortex sheet at the nodes in terms of tangential derivatives of the double-layer density (surface circulation).

14. `vs_3d_curl`
    Computation of the component of the curl of a vector function normal to the vortex sheet.

Input file:

1. `ldr_3d_ext.dat`
   Specification of the problem parameters.

Output file:

1. `ldr_3d_int.out`
   Computed values of the double-layer strength and normal derivative of the surface function f at the nodes.

References

[1] Pozrikidis, C. (2011) *Introduction to Theoretical and Computational Fluid Dynamics,* Second Edition, Oxford University Press.

Directory: lnm_3d

This directory contains a code that solves Laplace's equation in the interior or exterior of a closed three-dimensional surface, subject to the Neumann boundary condition that specifies the surface distribution of the normal derivative of the unknown function. The problem is formulated as an integral equation of the first kind originating from Green's third identity, using a boundary-element method. The mathematical formulation and numerical method are discussed in Reference [1].

References

[1] Pozrikidis, C. (2002) *A Practical Guide to Boundary-Element Methods with the Software Library BEMLIB.* Chapman & Hall/CRC.

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Fluid Dynamics Software Library